TRACT XXXVIII.

MISCELLANEOUS PRACTICAL PROBLEMS, &c, ILLUSTRATING SOME OF THE FOREGOING PRINCIPLES.

PROBLEM I.

It is required to find the Diameter of a Circular Parachute, by means of which a man of 150*lb weight may descend to the earth, from a Balloon at a height in the air, with the Velocity of only* 10 *feet in a second of time, being the Velocity acquired by a body freely descending through a space of only* 1 *foot* $6\frac{3}{4}$ *inches, or of a man jumping down from a height of* $18\frac{3}{4}$ *inches: the Parachute being made of such materials and thickness, that a circle of it of* 50 *feet diameter, weighs only* 150*lb, and so in proportion more or less according to the area of the circle.*

If a falling body descend with a uniform velocity, it must necessarily meet with a resistance, from the medium it descends in, equal to the whole weight that descends. Let x denote the diameter of the parachute, and $a = \cdot7854$; then ax^2 will be its area, and as $50^2 : x^2 :: 150 : \frac{3}{50}x^2$ the weight of the same, to which adding 150lb, the man's weight, the sum $\frac{3}{50}x^2 + 150$ will be the whole descending weight. Again, in the table of resistances, at pa. 189, Tract 36, art. 42, we find that a circle of $\frac{2}{9}$ of a square foot area, moving with 10 feet velocity, meets with a resistance of $\cdot57$ ounces$=\cdot0475$lb; and the resistances, with the same velocity, being as the surfaces, therefore as $\frac{2}{9} : \cdot0475 :: ax^2 : \cdot21375ax^2 = \cdot16788x^2$ the resistance of the air to the parachute, to which the descending weight must be equal; that is, $\cdot16788x^2 = \frac{3}{50}x^2 + 150$; hence $\cdot10788x^2 = 150$, or $x^2 = 1390\cdot5$, and hence $x = 37\frac{2}{7}$ feet, the diameter of the parachute required.

PROBLEM II.

To determine the Effects of Pile-Engines.

The form of the pile-engine, as used by the ancients, is not known. Many have been invented and described by the moderns. Among all these, that appears to be the best which was invented by M. Vauloue, as described by Dr. Desaguliers, and was used at piling the foundations at building Westminster Bridge. Its chief properties are, that the ram or weight be raised with the least expence of force, or with the fewest men; that it fall freely from its greatest height; and that, having fallen, it is presently laid hold of by the forceps, and so raised up to its height again. By which means, in the shortest time, and with the fewest men, or the least force, the most piles can be driven to the greatest depth.

Belidor has given some theory as to the effect of the pile-engine; but it appears to be founded on an erroneous principle: he deduces it from the laws of the collision of bodies. But who does not perceive that the rules of collision suppose a free motion and a non-resisting medium? It cannot therefore be applied in the present case, where a very great resistance is opposed to the pile by the ground. We shall therefore here endeavour to explain another theory of this machine.

Since the percussion of the weight acts on the pile during the whole time the pile is penetrating and sinking in the earth, by each blow of the ram, during which time its whole force is spent; it is manifest that the effect of the blow is of that nature, which requires its force to be estimated by the square of the velocity. But the square of the velocity acquired by the fall of the ram, is as the height it falls from; therefore the force of any blow will be as the height fallen through. But it is also more or less in proportion to the weight of the ram; consequently the effect or force of each blow must be directly in the compound ratio of both, viz, as aw, where w denotes the weight, and a the altitude it falls from; or it will be simply as the altitude a, when the weight w is constant.

Again, the force of the blow is opposed by the mass of the pile, and by the consistence of the earth penetrated by the point of the pile, and also by the friction of the earth against the sides of the pile that have penetrated below the surface. Consequently the effect of the blow, or the depth penetrated by the pile, will be inversely in the compound ratio of these three, viz, inversely as mtf, where m denotes the mass of the pile, t the tenacity or cohesion of the earth, and f the friction of the surface penetrated in the earth. But, in the same soil and with the same pile, m and t are both constant, in which case the depth of penetration will be inversely only as f the friction. On all accounts then the penetration will be as $\frac{aw}{mtf}$, or simply as $\frac{a}{f}$ only, for the same weight and pile and soil.

To determine the depth sunk by the Pile, at each stroke of the ram.

After a few strokes, so as to give the pile a little hold in the ground, to make it stand firmly, the blows of the ram may be considered as commencing, and causing the pile to sink a little at every stroke, by which small successive sinkings of the pile, the space the ram falls through will be successively increased by these small accessions, and the force of the successive blows proportionally increased. But these, on the other hand, are resisted and opposed by the friction of the part of the pile which has been sunk before, and which also sinks at each stroke; and as the quantities of these rubbing surfaces increase in a greater ratio to each other, than the heights fallen through, that is, the resisting forces increasing faster than the impelling forces, it is manifest that the depths successively sunk by the blows must gradually decrease by little and little every time; which is also found to be quite conformable to experience. Thus then the successive sinkings will proceed gradually diminishing, till they become so small as to be almost imperceptible.

Now it was found above that $\frac{a}{f}$ is as the penetration by any blow of the ram, by the same pile in the same soil, that is, as the height fallen directly, and as the resistance or friction in the earth inversely. Let a' denote any other and greater height, by an after stroke, and f' its friction; also p the penetration by the former blow, and p' that by the latter, which must be the smaller: then, by the foregoing principle, $\frac{a}{f} : \frac{a'}{f'} :: p : p'$; hence $a : a' :: pf : p'f'$, which is a general theorem.

But now, with respect to the quantity of friction from any blow, though it be not known from experiment that the friction is exactly proportional to the rubbing surface, there is great reason to believe that it must be at least very nearly so: there is also equal reason to conclude that the effect or resistance from that rubbing surface must be nearly or exactly as the length of space it moves over, that is, by the penetration of the pile by any blow. Now, if d denote the depth of the pile in the ground before any new blow is struck by the ram, and b the depth or penetration produced by the blow, then the length of the rubbing surface will be $d + \frac{1}{2}b$; for, the length of the rubbing surface is only d at the beginning of the motion, and it is $d + b$ at the end of it, the medium of the two, or $d + \frac{1}{2}b$, is therefore the due length of the surface, and the space or depth it moves over is b; therefore the whole resistance from the friction is $(d + \frac{1}{2}b)b$. If d' then denote any other depth of the pile in the earth, and b' the next penetration, then $(d' + \frac{1}{2}b')b'$ will be its friction. Substituting now b for p, and b' for p', also $d+\frac{1}{2}b$ for f, and $d'+\frac{1}{2}b'$ for f', in the general theorem $a : a' :: pf : p'f'$, it becomes $a : a' :: (d+\frac{1}{2}b)b : (d'+\frac{1}{2}b')b'$, for the general relation between the heights fallen and the resistance and penetration.

This theorem will very conveniently give the series of effects, or successive sinkings of the piles, by the blows of the ram. Thus, after the pile has been properly fixed, or indeed driven to any depth in the earth, denoted by d, then to give a blow, the ram falls from the height $a + d$, and thereby sinks the pile the space b suppose; hence, for the next stroke, the fall will be $a + d + b = a'$ in the theorem above, and $d' + \frac{1}{2}b' = d + b + \frac{1}{2}b'$, the next penetration or sinking being b'; theref. $a + d : a + d + b :: (d + \frac{1}{2}b)b : (d + b + \frac{1}{2}b')b'$, a proportion which gives the quadratic equa. $b'^2 + 2b'(d+b) = \frac{a+d+b}{a+d} \times (2d + b)b$, the root of which is $b' = -(d + b) + \sqrt{[(d + b)^2 + \frac{a+d+b}{a+d} \times (2d + b)b]} = \frac{a + d + b}{a + d} \times \frac{d + \frac{1}{2}b}{d + b} b$ nearly, or indeed $= \frac{d+\frac{1}{2}b}{d+b} b$ nearly, because b is small in comparison with $a + d$.

Now, for an example in numbers, suppose $a = 5$ feet $=$ 60 inches, $d = 10$, $b = 3$, that is $a = 60$ the height of the ram above the top of the pile before this enters the ground; $d = 10$, after being fixed in the ground; and $b = 3$ the sinking by the next blow: then $\frac{d + \frac{1}{2}b}{d + b} b = \frac{11 \cdot 5}{13} \times 3 = 2 \cdot 65 = b'$ the 2d stroke. Next, substituting $d + b$ for d, and b' for b, the same theorem gives 2·48 for the next sinking, or the next value of b'. And so on continually, by which means the series of the successive corresponding values of the letters will be as in the margin, the last column showing the several successive sinkings of the pile by the repeated strokes of the ram.

Specimen of the Series of the Successive values of d, b, b'.

d	b	b'
10	3	2·65
13	2·65	2·48
15·65	2·49	2·32
18·14	2·32	2·19
20·46	2·19	2·08
&c.		

Scholium. Thus then it appears, that the effect of any operation of pile-driving may be determined. It is manifest also that the greater a is, or the higher the top of the machine is where the ram falls from, above the top of the pile at first, the greater will be every stroke of the ram, and consequently the fewer the strokes requisite to drive the pile to the requisite depth. But then every stroke will take a longer time, as the ram will be both longer in falling and longer in raising: so that it may be a question whether, on the whole, the business may be effected in the less time by a greater height of the machine, or whether there be any limit to the height, so as to produce the greatest effect in a given time.

To answer this question, let x denote the indeterminate height from which any weight w is to fall, z the time of raising it after a fall, which time is supposed to be as the height x to which it is raised, also m the given time of producing a proposed effect; then $\frac{1}{4}\sqrt{x}$ = the time of the weight falling; therefore $\frac{1}{4}\sqrt{x} + z$ = the whole time of one stroke; conseq. $\frac{m}{\frac{1}{4}\sqrt{x} + z}$ or $\frac{4m}{\sqrt{x} + 4z}$ is the number of strokes made in the given time m, and hence $\frac{4mxw}{\sqrt{x} + 4z}$ = the whole force or effect in the time m. Now this effect or fraction increases continually as x increases, because the numerator increases faster than the denominator, since the former increases as x, while, in the latter, though the one term z increases as x, yet the other term $\sqrt{x}$ only increases as the root of x. So that, on the whole, it appears that the effect, in any given time, increases more and more as the height is increased.

PROBLEM III.

To determine how far a man, who pushes with the force of 100*lb, can thrust a sponge into a piece of ordnance, whose diameter is* 5 *inches, and length* 10 *feet, when the barometer stands at* 30 *inches: the vent, or touch-hole, being stopped, and the sponge having no windage, that is, fitting the bore quite close?*

A column of quicksilver 30 inches high, and 5 in diameter, is $5^2 \times 30 \times \cdot7854 = 589\cdot05$ inches; which, at 8·102 oz each inch, weighs 4772·48 oz, or 298·28lb, which is the pressure of the atmosphere alone, being equal to the elasticity of the air in its natural state; to this adding the 100lb, gives 398·28lb, the whole external pressure. Then, as the spaces which a quantity of air possesses, under different pressures, are in the reciprocal ratio of those pressures, it will be, as 398·28 : 298·28 : : 10 feet or 120 inches : 90 inches nearly, the space occupied by the air; theref. $120-90=30$ inches, is the distance sought.

PROBLEM IV.

To assign the Cause of the Deflection of Military Projectiles.

It having been surmized that, in the practice of artillery, the deflection of the shot in its flight, to the right or left, from the line or direction the gun is laid in, chiefly arises from the motion of the gun during the time the shot is passing out of the piece: it is required then to determine what space an 18-pounder will recoil or fly back, while the shot is passing out of the gun; supposing its weight to be 4800lb, that of the carriage 2400lb, the quantity of powder 8lb, the length of the cylinder 108 inches, that of the charge 13 inches, and the diameter of the bore 5·13 inches; supposing also that the resistance from the friction between the platform and carriage is equal to 3600lb?

It is well known that confined gunpowder, when fired, immediately changes in a great measure into an elastic air, which endeavours to expand in all directions. Now, in the question, the action of this fluid is exerted equally on the bottom of the bore of the gun and on the ball, during the passage of the latter through the cylinder; the two bodies therefore move in opposite directions, with velocities which are at all times in the inverse ratio of the quantities of matter moved. Now let x be the space through which the gun recoils; then, as the charge occupies 13 inches of the barrel, and the semidiameter of the barrel is 2·565, the space moved through by the ball when it quits the piece, is $108 - 13 - 2{\cdot}565 - x = 92{\cdot}435 - x$: and as the elastic fluid expands in both directions, the quantity which advances towards the muzzle, is to that which retreats from it, as $92{\cdot}435-x$ to x: consequently $\frac{8x}{92{\cdot}435}$ and $\frac{92{\cdot}435-x}{92{\cdot}435} \times 8$ are the quantities of the powder which move, the former with the gun, and the latter with the ball; besides these, the weight of ball that moves forwards being 18lb, and of the weights and resistance backwards $4800 + 2400 + 3600 = 10800$lb, hence the whole weights moved in the two directions are $10800 + \frac{8x}{92\cdot435}$ and $18 + \frac{92\cdot435-x}{92\cdot435} \times 8$, or $\frac{998298+8x}{92\cdot435}$ and $\frac{2403\cdot31-8x}{92\cdot435}$, or as the numerators of these only. But when the time and moving force are given, or the same, then the spaces are inversely as the quantities of matter; therefore $x : 92\cdot435 - x : : 2403\cdot31 - 8x : 998298 + 8x$, or by composition, $x : 92\cdot435 : : 2403\cdot31 - 8x : 1000701\cdot31$, and by div. $x : 1 : : 2403\cdot31 - 8x : 10826$, theref. $10826x = 2403\cdot31 - 8x$, or $10834x = 2403\cdot31$, and hence $x = \cdot2218$ inch $= \frac{2}{9}$ of an inch nearly, or the recoil of the gun is less than a quarter of an inch.

Hence it may be concluded, that so small a recoil, straight backwards, can have no effect in causing the ball to deviate from the pointed line of direction: and that it is very probable we are to seek for the cause of this effect in the ball striking or rubbing against the sides of the bore, in its passage through it, especially near the exit at the muzzle; by which it must happen, that if the ball strike against the right side, the ball will deviate to the left; if it strike on the left side, it must deviate to the right; if it strike against the under side, it must throw the ball upwards, and make it to range farther; but if it strike against the upper side, it must beat the ball downwards, and cause a shorter range: all which irregularities are found to take place, especially in guns that have much windage, or which have the balls too small for the bore.

PROBLEM V.

A Ball of Lead, of 4 inches diameter, being dropped from the top of a tower, of 65 yards high, falls into a cistern full of water at the bottom of the tower, of $20\frac{1}{4}$ yards deep: it is required to determine the times of falling, both to the surface and to the bottom of the water.

The fall in air is 195 feet, and in water $60\frac{3}{4}$ feet. By the common rules of descent, as $\sqrt{16} : \sqrt{195} :: 1'' : \frac{1}{4}\sqrt{195} =$ 3·49 seconds, the time of descending in air. And as $\sqrt{16} : \sqrt{195} :: 32 : 8\sqrt{195} = 111{\cdot}71$ feet, the velocity at the end of that time, or with which the ball enters the water.

Again, by prob. 22 of vol. 2, art. 2 of the Course, the space $s = \frac{1}{2b} \times$ hyp. log. of $\frac{a-e^2}{a-v^2}$, or rather $\frac{1}{2b} \times$ hyp. log. of $\frac{e^2-a}{v^2-a}$ (the velocity being decreasing, and e^2 greater than a) $=$ $\frac{m}{2b} \times$ com. log. of $\frac{e^2-a}{v^2-a}$, where $N = 11325$ the density of lead, $n = 1000$ that of water, $a = \frac{256d(N-n)}{3n}$, $b = \frac{3n}{8dN}$, $e =$ 111·71 the velocity at entering the water, and v the velocity at any time afterwards, also d the diameter of the ball $= 4$ inches, and $m = 2{\cdot}302585$ the hyp. log. of 10.

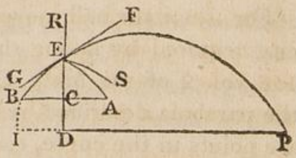
Hence then $N = 11325$, $n = 1000$, $N - n = 10325$, $d = \frac{4}{12} = \frac{1}{3}$; then $a = \frac{256d(N-n)}{3n} = \frac{256 \cdot 10325}{9000} = 293\frac{1}{2}$, and $b = \frac{3n}{8dN} = \frac{9n}{8N} = \frac{9000}{90600} = \frac{15}{151} = \frac{1}{10}$ nearly. Also $e = 111{\cdot}71$; therefore $s = 60\frac{3}{4} = \frac{m}{2b} \times$ log. of $\frac{e^2-a}{v^2-a} = 5m \times$ log. $\frac{e^2-a}{v^2-a}$. This theorem will give s when v is given, and by reverting it will give v in terms of s in the following manner.

Dividing by $5m$ gives $\frac{s}{5m} =$ log. of $\frac{e^2-a}{v^2-a} = ns$, by putting $n = \frac{1}{5m}$; therefore, the natural number is $10^{ns} = \frac{e^2-a}{v^2-a}$; hence $v^2 - a = \frac{e^2-a}{10^{ns}}$, and $v = \sqrt{(a + \frac{e^2-a}{10^{ns}})}$, which, by substituting the numbers above mentioned for the letters, gives $v = 17{\cdot}134$ for the last velocity, when the space $s = 60\frac{3}{4}$, or when the ball arrives at the bottom of the water.

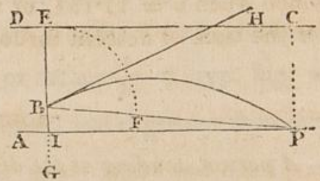
But now to find the time of passing through the water, putting $t =$ any time in motion, and s and v the corresponding space and velocity, the general theorem for variable forces gives $\dot{t} = \frac{\dot{s}}{v}$. But the above general value of s being $\frac{1}{2b} \times$ hyp. log. $\frac{e^2 - a}{v^2 - a}$ or $5 \times$ hyp. log. $\frac{e^2 - a}{v^2 - a}$, therefore its fluxion $\dot{s} = \frac{-10v\dot{v}}{v^2 - a}$, consequently, $\dot{t}$ or $\frac{\dot{s}}{v} = \frac{-10\dot{v}}{v^2 - a}$, the correct fluent of which is $\frac{5}{\sqrt{a}} \times$ hyp. log. $(\frac{e - \sqrt{a}}{e + \sqrt{a}} \times \frac{v + \sqrt{a}}{v - \sqrt{a}}) = t$ the time, which when $v = 17{\cdot}134$, or $s = 60\frac{3}{4}$, gives $2{\cdot}6542$ seconds, for the time of descent through the water.

PROBLEM VI.

A person standing at the distance of 10 *feet from the bottom of a wall, which is supposed perfectly smooth and hard, desires to know in what direction he must throw an elastic ball against it, with a velocity of* 80 *feet per second, so that, after reflection from the wall, it may fall at the greatest distance possible from the bottom, on the horizontal plane, which is* $2\frac{1}{2}$ *feet below the hand discharging the ball?*



In the annexed figure let DR be the wall against which the ball is thrown, from the point A, in such a direction, that it shall describe the parabolic curve AE before striking the wall, and afterwards be so reflected as to describe the curve EP. Now if ES be the tangent at the point E, to the curve AE described before the reflection, and EF the tangent at the same point to the curve which the ball will describe after reflection, then will the angle REF be = CES; and if the curve PE be produced, so as to have GF for its tangent, it will meet AC produced in B, making BC = AC, and the curve AE will be similar and equal to the portion BE of the parabola BEP, but turned the contrary way. Conceiving either the two curves AE and EP, or the continued curve BEP, to be described by a projectile in its motion, it is manifest that, whether the greater portion of the curve be described before or after the ball reaches the wall DR, will depend on its initial velocity, and on the distance AC or BC, and on the angle of projection. The problem then is now reduced to this, viz, To find the angle at which a ball shall be projected from B, with a given impetus, so that the distance DP, at which it falls, from the given point D, on the plane DP, parallel to the horizon, shall be a maximum.



Now this problem may be constructed in the following manner: From any point E in the horizontal line DC, let fall the indefinite perp. EG, on which set off $EB =$ the impetus corresponding to the given velocity, and $BI = 2\frac{1}{2}$ the distance of the horizontal plane below the point of projection; also, through I draw AP parallel to DC. From the point B set off $BP = BE + EI$, and bisect the angle EBP by the line BH: then will BH be the required direction of the ball, and IP the maximum range on the plane AP.

For, since the ball moves from the point B, with the velocity acquired by falling through EB, it is manifest, from p. 156 vol. 2 of the Math. Course, that DC is the directrix of the parabola described by the ball. And since both B and P are points in the curve, each of them must, from the nature of the parabola, be as far from the focus as it is from the directrix; therefore B and P will be the greatest distance from each other when the focus F is directly between them, that is, when $BP = BE + CP$. And when BP is a maximum, since BI is constant, it is obvious that IP is a maximum also. Further, the angle FBH being $=$ EBH, the line BH is a tangent to the parabola at the point B, and consequently it is the direction necessary to give the range IP.

Cor. 1. When B coincides with I, IP will be $=$ BP $=$ BE $+$ EI $=$ 2EI, and the angle EBH will be $45°$; as is also manifest from the common modes of investigation.

Cor. 2. When the impetus corresponding to the initial velocity of the ball is very great compared with AC or BC (fig. 1), then the part AE of the curve will very nearly coincide with its tangent, and the direction and velocity at A may be accounted the same as those at E without any sensible error. In this case too the impetus BE (fig. 2) will be very great compared with BI, and consequently, B and I nearly coinciding, the angle EBH will differ but little from $45°$.

Calcul. From the foregoing construction the calculation will be very easy. Thus, the first velocity being 80 feet$=v$, then (vol. 2 pa. 156 of the Course) $\frac{v^2}{4g} = \frac{80 \times 80}{64\frac{1}{3}} = 99{\cdot}48186$ $=$ BE the impetus; hence EI $=$ FP $= 101{\cdot}98186$, and BP $=$ BE $+$ EI $= 201{\cdot}46372$. Now, in the right-angled triangle BIP, the sides BI and BP are known, hence IP $= 201{\cdot}4482$, and the angle IBP $= 89° \; 17' \; 20''$: half the suppl. of this angle is $45° \; 21' \; 20''$ $=$ EBH. And, in fig. 1, IP $-$ ID $= 201{\cdot}4482$ $- 10 = 191{\cdot}4482 =$ DP, the distance the ball falls from the wall after reflection.

PROBLEM VII.

From what height above the given point A *must an elastic ball be suffered to descend freely by gravity, so that, after striking the hard plane at* B, *it may be reflected back again, to the point* A, *in the least time possible from the instant of dropping it?*

Let C be the point required; and put AC$=x$, and AB $= a$; then $\frac{1}{4}\sqrt{}$CB $= \frac{1}{4}\sqrt{(a+x)}$ is the time in CB, and $\frac{1}{4}\sqrt{}$CA $= \frac{1}{4}\sqrt{x}$ is the time in CA; therefore $\frac{1}{4}\sqrt{(a+x)} - \frac{1}{4}\sqrt{x}$ is the time down AB, and the time of rising from B to A again: hence the whole time of falling through CB and returning to A, is $\frac{1}{2}\sqrt{(a+x)}$ $-\frac{1}{4}\sqrt{x}$, which must be a min. or $2\sqrt{(a+x)} - \sqrt{x}$ a minimum, in fluxions $\frac{\dot{x}}{\sqrt{(a+x)}} - \frac{\dot{x}}{2\sqrt{x}} = 0$, and hence $x = \frac{1}{3}a$, that is, $AC = \frac{1}{3}AB$, gives the point C fallen from.

PROBLEM VIII.

A cylinder of oak is depressed in water till its top is just level with the surface, and then is suffered to ascend; it is required to determine the greatest altitude to which it will rise, and the time of its ascent.

Let $a =$ the length, and b the area or base of the cylinder, m the specific gravity of oak, that of water being 1, also x any variable height through which the cylinder has ascended. Then, $a - x$ being the part still immersed in the water, $(a - x) \times b \times 1 = (a - x)b$ is the force of the water upwards to raise the cylinder; and $a \times b \times m = abm$ is the weight of the cylinder opposing its ascent; therefore the efficacious force to raise the cylinder is $(a - x)b - abm$; and, the mass being abm, the accelerating force is

$$\frac{(a-x)b - abm}{abm} = \frac{a-x-am}{am} = \frac{an-x}{am} = f,$$

putting $n = 1 - m$ the difference between the specific gravities of water and oak.

Now if v denote the velocity of ascent at the same time when x space is ascended, then by the theorems for variable forces, $v\dot{v} = 32f\dot{x} = \frac{32}{am} \times (an\dot{x} - x\dot{x})$, therefore $v^2 = \frac{32}{am} \times (2anx - x^2)$, and $v = 8\sqrt{\frac{2anx - x^2}{2am}}$: but when the cylinder has acquired its greatest ascent, v and $v^2 = 0$, therefore then $2anx - x^2 = 0$, and hence $x = 2an$ the part of the cylinder that rises out of the water, being $= \cdot 15a$ or $\frac{3}{20}$ of its length.

To find when the velocity is the greatest, the factor $2anx - x^2$ in the velocity must be a maximum, then $2an\dot{x} - 2x\dot{x} = 0$, and $x = an$, being the height above the water when the velocity is the greatest, and which it appears is just equal to the half of $2an$ above found for the greatest rise, when the upward motion ceases, and the cylinder descends again to the same depth as at first, after which it again returns ascending as before; and so on, continually playing up and down to the same highest and lowest points, like the vibrations of a pendulum, the motion ceasing in both cases in a similar manner at the extreme points, then returning, it gradually accelerates till arriving at the middle point, where it is the greatest, then gradually retarding all the way to the next extremity of the vibration, thus making all the vibrations in equal times, to the same extent between the highest and lowest points, except that, by the small tenacity and friction &c, of the water against the sides of the cylinder, it will be gradually and slowly retarded in its motion, and the extent of the vibrations decrease till at length the cylinder, like the pendulum, come to rest in the middle point of its vibrations, where it naturally floats in its quiescent state, with the part na of its length above the water.

The quantity of the greatest velocity will be found, by substituting na for x, in the general value of the velocity $8\sqrt{\frac{2anx-x^2}{2am}}$, when it becomes $8n\sqrt{\frac{a}{2m}} = \frac{4}{9}\sqrt{a}$ very nearly, the value of m being $\cdot 925$, and consequently that of $n = 1 - m = \cdot 075$.

To find the time t answering to any space x. Here $\dot{t} = \frac{\dot{x}}{v} = \frac{\dot{x}}{8\sqrt{\frac{2nax-x^2}{2ma}}} = \sqrt{\frac{ma}{32}} \times \frac{\dot{x}}{\sqrt{(2nax-x^2)}}$, and by the 13th form the fluent is $t = \frac{1}{8}\sqrt{2ma} \times \text{A}$, where A denotes the circular arc to radius 1 and versed sine $\frac{x}{na}$. Now at the middle of a vibration x is $= na$, and then the vers. $\frac{x}{na} = \frac{na}{na} = 1$ the radius, and A is the quadrantal arc $= 1\cdot 5708$; then the flu. becomes $\frac{1}{8}\sqrt{2ma} \times 1\cdot 5708 = \cdot 17\sqrt{a} \times 1\cdot 5708 = \cdot 267\sqrt{a}$ for the time of a semivibration; hence the time of each whole vibration is $\cdot 534\sqrt{a} = \frac{8}{15}\sqrt{a}$, which time therefore depends on the length of the cylinder a. To make this time $= 1$ second, a must be $= (\frac{15}{8})^2$ very nearly $= 3\frac{1}{2}$ feet, or 42 inches. That is, the oaken cylinder of 42 inches length makes its vertical vibrations each in 1 second of time, or is isochronous with a common pendulum of $39\frac{1}{8}$ inches long, the extent of each vibration of the former being $6\frac{3}{10}$ inches.

PROBLEM IX.

Required to determine the quantity of matter in a sphere, the density varying as the nth power of the distance from the centre?

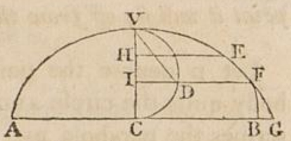
Let r denote the radius of the sphere, d the density at its surface, $a = 3{\cdot}1416$ the area of a circle whose radius is 1, and x any distance from the centre. Then $4ax^2$ will be the surface of a sphere whose radius is x, which may be considered by expansion as generating the magnitude of the solid; therefore $4ax^2\dot{x}$ will be the fluxion of the magnitude; but as $r^n : x^n :: d : \frac{dx^n}{r^n}$ the density at the distance x, therefore $4ax^2\dot{x} \times \frac{dx^n}{r^n} = \frac{4adx^{n+2}\dot{x}}{r^n}$ = the fluxion of the mass, the fluent of which $\frac{4adx^{n+3}}{(n+3)r^n}$, when $x = r$, is $\frac{4adr^3}{n+3}$, the quantity of the matter in the whole sphere.

Corol. 1. The magnitude of a sphere whose radius is r, being $\frac{4}{3}ar^3$, which call m; then the mass or solid content will be $\frac{3d}{n+3} \times m$, and the mean density is $\frac{3d}{n+3}$.

Corol. 2. It having been computed, from actual experiments, that the medium density of the whole mass of the earth is about 5 times the density d at the surface, we can now determine what is the exponent of the decreasing ratio of the density from the centre to the circumference, supposing it to decrease by a regular law, viz, as r^n; for then it will be $5d = \frac{3d}{n+3}$, and hence $n = -\frac{12}{5}$. So that, in this case, the law of decrease is as $r^{-\frac{12}{5}}$, or as $\frac{1}{r^{\frac{12}{5}}}$, that is, inversely as the $\frac{12}{5}$ths power of the radius.

PROBLEM X.

Required to determine where a body, moving down the convex side of a cycloid, will fly off and quit the curve.



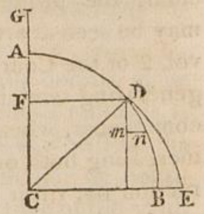
Let AVEB represent the cycloid, the properties of which may be seen at arts. 146 and 147 vol. 2 of the Course, and VDC its generating semicircle. Let E be the point where the motion commences, whence it moves along the curve, its velocity increasing both on the curve, and also in the horizontal direction DF, till it come to such a point, F suppose, that the velocity in the latter direction is become a constant quantity, then that will be the point where it will quit the cycloid, and afterwards describe a parabola FG, because the horizontal velocity in the latter curve is always the same constant quantity, by art. 76 vol. 2 of the Academy Course.

Put the diameter VC $= d$, VH $= a$, VI $= x$; then VD $= \sqrt{dx}$, and ID $= \sqrt{(dx - x^2)}$. Now the velocity in the curve at F, in descending down EF, being the same as by falling through HI or $x - a$, will be $= 8\sqrt{(x - a)}$; but this velocity in the curve at F, is to the horizontal velocity there, as VD to ID, because VD is parallel to the curve or to the tangent at F, that is $\sqrt{dx} : \sqrt{(dx - x^2)} :: 8\sqrt{(x - a)} : \frac{8\sqrt{(x - a)} \times \sqrt{(d - x)}}{\sqrt{d}}$, which is the horizontal velocity at F, where the body is supposed to have that velocity a constant quantity; therefore also $\sqrt{(x - a)} \times \sqrt{(d - x)}$, as well as $(x - a) \times (d - x) = ax + dx - ad - x^2$ is a constant quantity, and also $ax + dx - x^2$; but the fluxion of a constant quantity is equal to nothing, that is $a\dot{x} + d\dot{x} - 2x\dot{x} = 0 = a + d - 2x$, and hence $x = \frac{1}{2}a + \frac{1}{2}d$ $=$ VI, the arithmetical mean between VH and VC.

If the motion should commence at V, then x or VI would be $= \frac{1}{2}d$, and I would be the centre of the semicircle.

PROBLEM XI.

If a body begin to move from A, *with a given velocity, along the quadrant of a circle* AB; *it is required to show at what point it will fly off from the curve.*



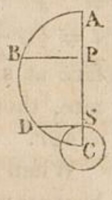
Let D denote the point where the body quits the circle ABD, and then describes the parabola DE. Draw the ordinate DF, and let GA be the height producing the velocity at A. Put GA $= a$, AC or CD $= r$, AF $= x$; then the velocity in the curve at D will be the same as that acquired by falling through GF or $a + x$, which is, as before, $8\sqrt{(a + x)}$; but the velocity in the curve is to the horizontal velocity as Dn to mn or as CD to CF by similar triangles, that is, as $r : r - x :: 8\sqrt{(x + a)} : 8\sqrt{(x + a)} \times \frac{r - x}{r}$, which is to be a constant quantity where the body leaves the circle, therefore also $(r - x)\sqrt{(x + a)}$ and $(r - x)^2 \times (x + a)$ a constant quantity; the fluxion of which made to vanish, gives $x = \frac{r - 2a}{3} =$ AF.

Hence, if $a = 0$, or the body only commence motion at A, then $x = \frac{1}{3}r$, or AF $= \frac{1}{3}$AC when it quits the circle at D. But if a or GA were $= \frac{1}{2}r$ or $\frac{1}{2}$AC, then $r - 2a = 0$, and the body would instantly quit the circle at the vertex A, and describe a parabola circumscribing it, and having the same vertex A.

PROBLEM XII.

The force of attraction, above the earth, being inversely as the square of the distance from the centre; it is proposed to determine the time, velocity, and other circumstances, attending a heavy body falling from any given height; the descent at the earth's surface being $16\frac{1}{12}$ *feet, or* 193 *inches, in the first second of time.*

Put



$r =$ CS the radius of the earth,
$a =$ CA the dist. fallen from,
$x =$ CP any variable distance,
$v =$ the velocity at P,
$t =$ time of falling there, and
$g = 16\frac{1}{12}$, half the veloc. or force at S,
$f =$ the force at the point P.

Then we have the three following equations, viz.

$x^2 : r^2 :: 1 : f = \frac{r^2}{x^2}$ the force at P, when the force of gravity is considered as 1;

$\dot{t}v = -\dot{x}$, because x decreases; and

$v\dot{v} = -2gf\dot{x} = -\frac{2gr^2\dot{x}}{x^2}$.

The fluents of the last equation give $v^2 = \frac{4gr^2}{x}$. But when $x = a$, the velocity $v = 0$; therefore, by correction, $v^2 = \frac{4gr^2}{x} - \frac{4gr^2}{a} = 4gr^2 \times \frac{a-x}{ax}$; or $v = \sqrt{(\frac{4gr^2}{a} \times \frac{a-x}{x})}$, a general expression for the velocity at any point P.

When $x = r$, this gives $v = \sqrt{(4gr \times \frac{a-r}{a})}$ for the greatest velocity, or the velocity when the body strikes the earth.

When a is very great in respect of r, the last velocity becomes $(1 - \frac{r}{2a}) \times \sqrt{4gr}$ very nearly, or nearly $\sqrt{4gr}$ only, which is accurately the greatest velocity by falling from an infinite height. And this, when $r = 3965$ miles, is 6·9506 miles per second. Also, the velocity acquired in falling from the distance of the sun, or 12000 diameters of the earth, is 6·9505 miles per second. And the velocity acquired in falling from the distance of the moon, or 30 diameters, is 6·8927 miles per second.

Again, to find the time; since $\dot{t}v = -\dot{x}$, therefore $\dot{t} = \frac{-\dot{x}}{v} = \sqrt{\frac{a}{4gr^2}} \times \frac{-x\dot{x}}{\sqrt{ax-xx}}$; the correct fluent of which gives $t = \sqrt{\frac{a}{4gr^2}} \times (\sqrt{ax-xx} + \text{arc to diameter } a \text{ and vers. } a-x)$; or the time of falling to any point P =

$\frac{1}{2r}\sqrt{\frac{a}{g}} \times (\text{AB} + \text{BP})$. And when $x = r$, this becomes $t = \frac{1}{2}\sqrt{\frac{a}{g}} \times \frac{\text{AD}+\text{DS}}{\text{SC}}$ for the whole time of falling to the surface at S; which is evidently infinite when a or AC is infinite, though the velocity is then only the finite quantity $\sqrt{4gr}$.

When the height above the earth's surface is given $= g$; because r is then nearly $= a$, and AD nearly $=$ DS, the time t for the distance g will be nearly - - - - - - - - - $\sqrt{\frac{1}{4gr^2}} \times 2\text{DS} = \sqrt{\frac{1}{4gr}} \times \sqrt{4gr} = 1''$, as it ought to be.

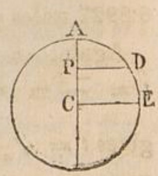
If a body, at the distance of the moon at A, fall to the earth's surface at S. Then $r = 3965$ miles, $a = 60r$, and $t = 416806'' = 4$ da. 19 h. 46′ 46″, which is the time of falling from the moon to the earth.

When the attracting body is considered as a point C; the whole time of descending to C will be - - - - - - - $\frac{1}{2r}\sqrt{\frac{a}{g}} \times \text{ABDC} = \frac{\cdot 7854a}{r}\sqrt{\frac{a}{g}} = \frac{10a}{51r}\sqrt{a} = \frac{\cdot 7854}{r}\sqrt{\frac{a^3}{g}}$.

Hence, the times employed by bodies, in falling from quiescence to the centre of attraction, are as the square roots of the cubes of the heights from which they respectively fall.

PROBLEM XIII.

The force of attraction below the earth's surface being directly as the distance from the centre; it is proposed to determine the circumstances of velocity, time, and space fallen by a heavy body from the surface, through a perforation made straight to the centre of the earth; abstracting from the effect of the earth's rotation, and supposing it to be a homogeneous sphere of 3965 miles radius.



Put $r =$ AC the radius of the earth,
$x =$ CP the dist. from the centre,
$v =$ the velocity at P,
$t =$ the time there,
$g = 16\frac{1}{12}$, half the force at A,
$f =$ the force at P.

Then CA : CP : : 1 : f; and the three equations are $rf = x$, and $v\dot{v} = -2gf\dot{x}$, and $\dot{t}v = -\dot{x}$. Hence $f = \frac{x}{r}$, and $v\dot{v} = \frac{-2gx\dot{x}}{r}$; the correct fluent of which gives $v = \sqrt{(2g \times \frac{r^2 - x^2}{r})}$ $= \text{PD}\sqrt{\frac{2g}{r}} = \text{PD}\sqrt{\frac{2g}{\text{CE}}}$, the velocity at the point P; where PD and CE are perpendicular to CA. So that the velocity at any point P, is as the perpendicular PD at that point.

When the body arrives at C, then $v = \sqrt{2gr} = \sqrt{(2g.\text{AC})}$ $= 25950$ feet or 4·9148 miles per second, which is the greatest velocity, or that at the centre C.

Again, for the time, $\dot{t} = \frac{-\dot{x}}{v} = \sqrt{\frac{r}{2g}} \times \frac{-\dot{x}}{\sqrt{(r^2 - x^2)}}$, and the fluents give $t = \sqrt{\frac{r}{2g}} \times$ arc to cosine $\frac{x}{r} = \sqrt{\frac{1}{2gr}} \times$ arc AD. So that the time of descent to any point P, is as the corresponding arc AD.

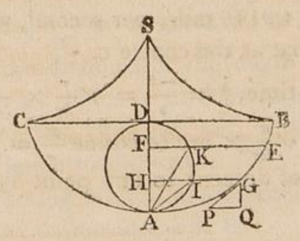
When P arrives at C, the above becomes $t =$ $\sqrt{\frac{1}{2gr}} \times$ quadrant $\text{AE} = \frac{\text{AE}}{\text{AC}}\sqrt{\frac{r}{2g}} = 1{\cdot}5708\sqrt{\frac{r}{2g}} = 1267\frac{1}{4}$ seconds $= 21'\ 7''\frac{1}{4}$, for the time of falling to the centre C.

The time of falling to the centre is the same quantity $1{\cdot}5708\sqrt{\frac{r}{2g}}$, from whatever point in the radius AC the body begins to move. For let n be any given distance from C at which the motion commences: then, by correction, $v =$ $\sqrt{[\frac{2g}{r}(n^2 - x^2)]}$; and hence $\dot{t} = \sqrt{\frac{r}{2g}} \times \frac{-\dot{x}}{\sqrt{(n^2 - x^2)}}$, the fluents of which give $t = \sqrt{\frac{r}{2g}} \times$ arc to cosine $\frac{x}{n}$; which, when $x = 0$, gives $t = \sqrt{\frac{r}{2g}} \times$ quadrant $= 1{\cdot}5708\sqrt{\frac{r}{2g}}$ for the time of descent to the centre C, the same as before.

As an equal force, acting in contrary directions, generates or destroys an equal quantity of motion, in the same time; it follows that, after passing the centre, the body will just ascend to the opposite surface at B, in the same time in which it fell to the centre from A; then from B it will return again in the same manner, through C to A; and so vibrate continually between A and B, the velocity being always equal at equal distances from C on both sides; and the whole time of a double oscillation, or of passing from A and arriving at A again, will be quadruple the time of passing over the radius AC, or $= 2 \times 3{\cdot}1416\sqrt{\frac{r}{2g}} = 1^{h}\,24'\,29''$.

PROBLEM XIV.

To find the Time of a Pendulum vibrating in the Arc of a Cycloid.



Let S be the point of suspension,

SA = the arc SB or SC the length of the pendulum,

CA = AB = SB or SC the semi-cycloid,

AD = DS the diameter of its generating circle, to which FKE, HIG are perpendiculars.

To any point G draw the tangent GP, also draw GQ parallel and PQ perpendicular to AD. Then PG is parallel to the chord AI by the nature of the curve. And, by the nature of forces, the force of gravity : force in direct. GP : : GP : GQ : : AI : AH : : AD : AI; in like manner, the force of grav. : force in curve at E : : AD : AK; that is, the accelerative force in the curve, is as the corresponding chord AI or AK of the circle, or as the arc AG or AE of the cycloid, since AG is always = 2AI. So that the process and conclusions for the velocity and time of describing any arc in this case, will be the same as in the last problem, the nature of the forces being the same, viz, as the distance to be passed over to the lowest point A.

From which it follows, that the time of a semi-vibration, in all arcs, AG, AE, &c, is the same constant quantity

$1{\cdot}5708\sqrt{\frac{r}{2g}} = 1{\cdot}5708\sqrt{\frac{A}{2g}} = 1{\cdot}5708\sqrt{\frac{l}{2g}}$, and the time of a whole vibration from B to C, or from C to B, is $3{\cdot}1416\sqrt{\frac{l}{2g}}$, where l = AS = AB is the length of the pendulum, $g = 16\frac{1}{12}$ feet, or 193 inches, and 3·1416 the circumference of a circle whose diameter is 1.

Since the time of a body's falling by gravity through $\frac{1}{2}l$, or half the length of the pendulum, is $\sqrt{\frac{l}{2g}}$, which being in proportion to $3{\cdot}1416\sqrt{\frac{l}{2g}}$, as 1 to 3·1416; therefore the diameter of a circle is to its circumference, as the time of falling through half the length of a pendulum, to the time of one vibration.

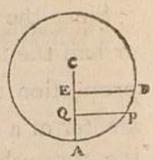
If the time of the whole vibration be 1 second, this equation arises, $1'' = 3{\cdot}1416\sqrt{\frac{l}{2g}}$, and hence $l = \frac{2g}{3{\cdot}1416^2} = \frac{g}{4{\cdot}9348}$, and $g = 3{\cdot}1416^2 \times \frac{1}{2}l = 4{\cdot}9348l$. So that if one of these, g or l, be given by experiment, these equations will give the other. When g, for instance, is supposed to be $16\frac{1}{12}$ feet, or 193 inches, then is $l = \frac{g}{4{\cdot}9348} = 39{\cdot}11$ the length of a pendulum to vibrate seconds. Or if $l = 39\frac{1}{8}$, the length of the seconds pendulum for the latitude of London, then is $g = 4{\cdot}9348l = 193{\cdot}07$ inches $= 16\frac{107}{1200}$ feet, or nearly $16\frac{1}{12}$ feet, for the space descended by gravity in the first second of time in the latitude of London, also agreeing with experiment.

Hence the times of vibration of pendulums, are as the square roots of their lengths; and the number of vibrations made in a given time, reciprocally as the square roots of the lengths. And hence also, the length of a pendulum vibrating n times in a minute, or 60″, is $l = 39\frac{1}{8} \times \frac{60^2}{n^2} = \frac{140850}{nn}$.

When a pendulum vibrates in a circular arc, as the length of the string is constantly the same, the time of vibration will be longer than in a cycloid; but the two times will approach nearer together as the circular arc is smaller; so that when it is very small, the times of vibration will be nearly equal. And hence $39\frac{1}{8}$ inches is the length of a pendulum vibrating seconds in the very small arc of a circle.

PROBLEM XV.

To find the Velocity and Time of a heavy Body descending down the Arc of a Circle, or vibrating in the Arc by a Line fixed in the Centre.



Let D be the beginning of the descent, C the centre, and A the lowest point of the circle; draw DE and PQ perpendicular to AC. Then the velocity in P being the same as in Q by falling through EQ, it will be $v = 2\sqrt{(g \times \text{EQ})} = 8\sqrt{(a - x)}$, when $a = \text{AE}$, $x = \text{AQ}$.

But the flux. of the time $\dot{t}$ is $= \frac{-\dot{\text{AP}}}{v}$, and $\dot{\text{AP}} = \frac{r\dot{x}}{\sqrt{(2rx - x^2)}}$ where r = the radius AC. Theref. $\dot{t} = \frac{r}{8} \times \frac{-\dot{x}}{\sqrt{(2rx - x^2)} \times \sqrt{(a - x)}}$ $= \frac{d}{16} \times \frac{-\dot{x}}{\sqrt{(ax - x^2)} \times \sqrt{(d - x)}} = \frac{-\sqrt{d}}{16} \times \frac{\dot{x}}{\sqrt{(ax - x^2)} \times \sqrt{(1 - \frac{x}{d})}}$, where $d = 2r$ the diameter.

Or $\dot{t} = \frac{-\sqrt{d}}{16} \times \frac{\dot{x}}{\sqrt{(ax - x^2)}} (1 + \frac{x}{2d} + \frac{1 \cdot 3x^2}{2 \cdot 4d^2} + \frac{1 \cdot 3 \cdot 5x^3}{2 \cdot 4 \cdot 6d^3} \&c)$, by developing $\sqrt{(1 - \frac{x}{d})}$ in a series.

But the fluent of $\frac{\dot{x}}{\sqrt{(ax - x^2)}}$ is $\frac{2}{a} \times$ arc to radius $\frac{1}{2}a$ and vers. x, or it is the arc whose rad. is 1 and vers. $\frac{2x}{a}$: which call A. And let the fluents of the succeeding terms, without the coefficients, be B, C, D, E, &c. Then will the flux. of any one, as $\dot{\text{Q}}$, at n distance from A, be $\dot{\text{Q}} = x^n\dot{\text{A}} = x\dot{\text{P}}$, which suppose also = the flux. of $b\text{P} - dx^{n-1}\sqrt{(ax - x^2)} = b\dot{\text{P}} - d(n - 1)\dot{x}x^{n-2}\sqrt{(ax - x^2)} - d\dot{x}x^{n-2} \times \frac{\frac{1}{2}ax - x^2}{\sqrt{(ax - x^2)}} = b\dot{\text{P}} - d\dot{x} \times \frac{(n - \frac{1}{2})ax^{n-1} - nx^n}{\sqrt{(ax - x^2)}} = b\dot{\text{P}} - d(n - \frac{1}{2})a\dot{\text{P}} + dnx\dot{\text{P}}$.

Hence, by equating the coefficients of the like terms, $d = \frac{1}{n}$; $b = \frac{2n - 1}{2n}a$; and $\text{Q} = \frac{(2n - 1)a\text{P} - 2x^{n-1}\sqrt{(ax - x^2)}}{2n}$. Which being substituted, the fluential terms become $-\frac{\sqrt{d}}{16} \times$ $(-\text{A} - \frac{1}{2d} \cdot \frac{a\text{A} - 2\sqrt{(ax - x^2)}}{2} - \frac{1 \cdot 3}{2 \cdot 4d^2} \cdot \frac{3a\text{B} - 2x\sqrt{(ax - x^2)}}{4} -$

$\frac{1 \, . \, 3 \, . \, 5}{2 \, . \, 4 \, . \, 6d^3} \cdot \frac{5ac - 2x^2 \sqrt{(ax - x^2)}}{6} -$ &c). Or the same fluents will be found by art. 32, pa. 238, vol. 3 of the Course.

But when $x = a$, those terms become barely $\frac{3 \cdot 1416 \sqrt{d}}{16} \times$ $(-1 - \frac{1^2 a}{2^2 d} - \frac{1^2 \, . \, 3^2 a^2}{2^2 \, . \, 4^2 d^2} - \frac{1^2 \, . \, 3^2 \, . \, 5^2 a^3}{2^2 \, . \, 4^2 \, . \, 6^2 d^3} -$ &c); which being subtracted, and x taken $= 0$, there arises for the whole time of descending down DA, or the corrected value of $t =$ $\frac{3 \cdot 1416 \sqrt{d}}{16} \times (1 + \frac{1^2 a}{2^2 d} + \frac{1^2 \, . \, 3^2 a^2}{2^2 \, . \, 4^2 d^2} + \frac{1^2 \, . \, 3^2 \, . \, 5^2 a^3}{2^2 \, . \, 4^2 \, . \, 6^2 d^3} +$ &c).

When the arc is small, as in the vibration of the pendulum of a clock, all the terms of the series may be omitted after the second, and then the time of a semi-vibration t is nearly $= \frac{1 \cdot 5708}{4} \sqrt{\frac{r}{2}} \times (1 + \frac{a}{8r})$. And theref. the times of vibration of a pendulum, in different arcs, are as $8r + a$, or 8 times the radius added to the versed sine of the arc.

If D be the degrees of the pendulum's vibration, on each side of the lowest point of the small arc, the radius being r, the diameter d, and $3 \cdot 1416 = p$; then is the length of that arc $\text{A} = \frac{pr\text{D}}{180} = \frac{pd\text{D}}{360}$. But the versed sine in terms of the arc is $a = \frac{\text{A}^2}{2r} - \frac{\text{A}^4}{24r^3} +$ &c $= \frac{\text{A}^2}{d} - \frac{\text{A}^4}{3d^3} +$ &c. Therefore $\frac{a}{d} = \frac{\text{A}^2}{d^2} - \frac{\text{A}^4}{3d^4} +$ &c $= \frac{p^2\text{D}^2}{360^2} - \frac{p^4\text{D}^4}{3.360^4} +$ &c, or only $= \frac{p^2\text{D}^2}{360^2}$ the first term, by rejecting all the rest of the terms on account of their smallness, or $\frac{a}{d} = \frac{a}{2r}$ nearly $= \frac{\text{D}^2}{13131}$. This value then being substituted for $\frac{a}{d}$ or $\frac{a}{2r}$ in the last near value of the time, it becomes $t = \frac{1 \cdot 5708}{4} \sqrt{\frac{r}{2}} \times (1 + \frac{\text{D}^2}{52524})$ nearly. And therefore the times of vibration in different small arcs, are as $52524 + \text{D}^2$, or as 52524 added to the square of the number of degrees in the arc.

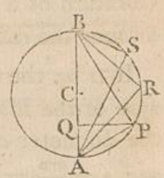
Hence it follows that the time lost in each second, by vibrating in a circle, instead of the cycloid, is $\frac{\text{D}^2}{52524}$; and consequently the time lost in a whole day of 24 hours, or $24 \times 60 \times 60$ seconds, is $\frac{5}{3}\text{D}^2$ nearly. In like manner, the seconds lost per day by vibrating in the arc of Δ degrees, is $\frac{5}{3}\Delta^2$.

Therefore, if the pendulum keep true time in one of these arcs, the seconds lost or gained per day, by vibrating in the other, will be $\frac{5}{3}(D^2 - \Delta^2)$. So, for example, if a pendulum measure true time in an arc of 3 degrees, it will lose $11\frac{2}{3}$ seconds a day by vibrating 4 degrees; and $26\frac{2}{3}$ seconds a day by vibrating 5 degrees; and so on.

And in like manner we might proceed for any other curve, as the ellipse, hyperbola, parabola, &c.

PROBLEM XVI.

To determine the Time of a Body descending down the Chord of a Circle.



Let C be the centre, AB the vertical diameter, AP any chord down which a body is to descend from P to A, and PQ perpendicular to AB. Now as the natural force of gravity in the vertical direction BA, is to the force urging the body down the plane PA, as the length of the plane AP, is to its height AQ; therefore the velocity in PA and QA, will be equal at all equal perpendicular distances below PQ; and consequently the

time in PA : time in QA : : PA : QA : : BA : PA; but
time in BA : time in QA : : $\sqrt{BA}$: $\sqrt{QA}$: : BA : PA;

hence, as three of the terms in each proportion are the same, the fourth terms must be equal, namely the time in BA = the time PA.

And in like manner the time in BP = the time in BA. So that, in general, the times of descending down all the chords BA, BP, BR, BS, &c, or PA, RA, SA, &c, are all equal, and each equal to the time of falling freely through the diameter. Which time is $\sqrt{\frac{2r}{g}}$, where $g = 16\frac{1}{12}$ feet, and $r =$ the radius AC; for $\sqrt{g} : \sqrt{2r} : : 1'' : \sqrt{\frac{2r}{g}}$.

Scholium. By comparing this with the results of the two preceding problems, it will appear that the times in the cycloid, and in the arc of a circle, and in any chord of the circle, are respectively as the three quantities

$$1,\ 1 + \frac{a}{8r}\ \&c, \text{ and } \frac{1}{\cdot 7854},$$

or nearly as the three quantities $1,\ 1 + \frac{a}{8r},\ 1\cdot 27324$; the first and last being constant, but the middle one, or the time in the circle, varying with the extent of the arc of vibration. Also the time in the cycloid is the least, but in the chord the greatest; for the greatest value of the series, in prob. 15, when $a = r$, or the arc AD is a quadrant, is 1·18014; and in that case the proportion of the three times is as the numbers 1, 1·18014, 1·27324. Moreover the time in the circle approaches to that in the cycloid, as the arc decreases, and they are very nearly equal when that arc is very small.

PROBLEM XVII.

To find the Time and Velocity of a Chain, consisting of very small links, descending from a smooth horizontal plane; the Chain being 100 *inches long, and* 1 *inch of it hanging off the Plane at the commencement of Motion.*

Put $a = 1$ inch, the length at the beginning;
$l = 100$ the whole length of the chain;
$x =$ any variable length off the plane.

Then x is the motive force to move the body,
and $\frac{x}{l} = f$ the accelerative force.

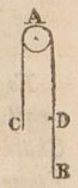
Hence $v\dot{v} = 2gf\dot{s} = 2g \times \frac{x}{l} \times \dot{x} = \frac{2gx\dot{x}}{l}$.

The fluents give $v^2 = \frac{2gx^2}{l}$. But $v = 0$ when $x = a$, theref. by correction, $v^2 = 2g \times \frac{x^2 - a^2}{l}$, and $v = \sqrt{(2g \times \frac{x^2 - a^2}{l})}$ the velocity for any length x. And when the chain just quits the plain, $x = l$, and then the greatest velocity is $\sqrt{(2g \times \frac{l^2 - a^2}{l})} = \sqrt{(2 \times 193 \times \frac{100^2 - 1^2}{100}} = \sqrt{\frac{386 \times 9999}{100}} =$ 196·45902 inches, or 16·371585 feet per second.

Again $\dot{t}$ or $\frac{\dot{s}}{v} = \sqrt{\frac{l}{2g}} \times \frac{\dot{x}}{\sqrt{(x^2-a^2)}}$; the correct fluent of which is $t = \sqrt{\frac{l}{2g}} \times \log. \frac{x + \sqrt{(x^2 - a^2)}}{a}$, the time for any length x. And when $x = l = 100$, it is $t = \sqrt{\frac{100}{386}} \times \log. \frac{100 + \sqrt{9999}}{1} = 2\cdot69676$ seconds, the time when the last of the chain just quits the plane.

PROBLEM XVIII.

To find the Time and Velocity of a Chain, of very small Links, quitting a Pulley, by passing freely over it: the whole Length being 200 Inches, and the one End hanging 2 Inches below the other at the beginning.



Put $a = 2$, $l = 200$, and $x =$ BD any variable difference of the two parts AB, AC. Then $\frac{x}{l} = f$, and $v\dot{v}$ or $2gf\dot{s} = 2g \cdot \frac{x}{l} \cdot \frac{1}{2}\dot{x} = \frac{gx\dot{x}}{l}$.

Hence the correct fluent is $v^2 = g \times \frac{x^2 - a^2}{l}$, and $v = \sqrt{(g \times \frac{x^2-a^2}{l})}$, the general expression for the velocity. And when $x = l$, or when C arrives at A, it is $v = \sqrt{(g \times \frac{l^2-a^2}{l})} = \sqrt{(193 \times \frac{200^2-2^2}{200})} = \sqrt{(386 \times \frac{100^2-1^2}{100})} = \sqrt{\frac{386 \times 9999}{100}} = 196\cdot45902$ inches, or $16\cdot371585$ feet for the greatest velocity when the chain just quits the pulley.

Again, $\dot{t}$ or $\frac{\dot{s}}{v} = \frac{\dot{x}}{2v} = \sqrt{\frac{l}{4g}} \times \frac{\dot{x}}{\sqrt{(x^2-a^2)}}$. And the correct fluent is $t = \sqrt{\frac{l}{4g}} \times \log. \frac{x + \sqrt{(x^2-a^2)}}{a}$, the general expression for the time. And when $x = l$, it becomes $t = \sqrt{\frac{l}{4g}} \times \log. \frac{l + \sqrt{(l^2-a^2)}}{a} = \sqrt{\frac{200}{772}} \times \log. \frac{200 + \sqrt{(200^2 - 2^2)}}{2} = \sqrt{\frac{100}{386}} \times \log. \frac{100 + \sqrt{9999}}{1} = 2\cdot69676$ seconds, the whole time when the chain just quits the pulley.

So that the velocity and time at quitting the pulley in this prob. and the plane in the last prob. are the same; the distance descended 99 being the same in both. For, though the weight l moved in this latter case, be double of what it was in the former, the moving force x is also double, because here the one end of the chain shortens as much as the other end lengthens, so that the space descended $\frac{1}{2}x$ is doubled, and becomes x; and hence the accelerative force $\frac{x}{l}$ or f is the same in both; and of course the velocity and time the same for the same distance descended.

PROBLEM XIX.

To find the Number of Vibrations made by two Weights, connected by a very fine Thread, passing freely over a Tack or a Pulley, while the less Weight is drawn up to it by the descent of the heavier Weight at the other End.

Suppose the motion to commence at equal distances below the pulley at B; and that the weights are 1 and 2 pounds.

Put a = AB, half the length of the thread;
$b = 39\frac{1}{8}$ inc. or $3\frac{25}{96}$ feet, the second's pend.
x = Bw = BW, any space passed over;
z = the number of vibrations.

Then $\frac{W-w}{W+w} = f = \frac{1}{3}$ is the accelerating force. And hence v or $\sqrt{4gfs} = \sqrt{4gfx}$, and $\dot{t}$ or $\frac{\dot{x}}{v} = \frac{\dot{x}}{\sqrt{4gfx}}$. But, by the nature of pendulums, $\sqrt{(a \pm x)} : \sqrt{b} :: 1$ vibr. : $\sqrt{\frac{b}{a \pm x}}$ the vibrations per second made by either weight, namely, the longer or shorter, according as the upper or under sign is used, if the threads were to continue of that length for 1 second. Hence then, as

$$1'' : \dot{t} :: \sqrt{\frac{b}{a \pm x}} : \dot{z} = \dot{t}\sqrt{\frac{b}{a \pm x}} = \sqrt{\frac{b}{4gf}} \times \frac{\dot{x}}{\sqrt{(ax \pm x^2)}},$$

the fluxion of the number of vibrations.

Now when the upper sign + takes place, the fluent is

$$z = 2\sqrt{\frac{b}{4gf}} \times \text{l.}\ \frac{\sqrt{x} + \sqrt{(a+x)}}{\sqrt{a}} = \sqrt{\frac{b}{4gf}} \times \text{l.}\ \frac{a + 2x + 2\sqrt{(ax + x^2)}}{a}.$$

And when $x = a$, the same then becomes $z = \sqrt{\frac{b}{gf}} \times$ log. $1 + \sqrt{2} = \sqrt{\frac{3b}{g}} \times$ log. $1 + \sqrt{2} = \sqrt{\frac{117\frac{3}{8}}{193}} \times$ log. $1 + \sqrt{2} =$ ·688511, the whole number of vibrations made by the descending weight.

But when the lower sign, or $-$, takes place, the fluent is $\sqrt{\frac{b}{4gf}} =$ arc to rad. 1 and vers. $\frac{2x}{a}$. Which, when $x = a$, gives $\frac{1}{2}p\sqrt{\frac{b}{gf}} = 3{\cdot}1416 \times \sqrt{\frac{3 \times 39\frac{1}{8}}{4 \times 193}} = \frac{3{\cdot}1416}{2} \times \sqrt{\frac{117\frac{3}{8}}{193}} =$ 1·227091, the whole number of vibrations made by the lesser or ascending weight.

Schol. It is evident that the whole number of vibrations, in each case, is the same, whatever the length of the thread is. And that the greater number is to the less, as 1·5708 to the hyp. log. of $1 + \sqrt{2}$.

Farther, the number of vibrations performed in the same time t, by an invariable pendulum, constantly of the same length a, is $\sqrt{\frac{b}{gf}} =$ ·781190. For, the time of descending the space a, or the fluent of $\dot{t} = \frac{\dot{x}}{\sqrt{4gfx}}$, when $x = a$, is $t = \sqrt{\frac{a}{gf}}$. And, by the nature of pendulums, $\sqrt{a} : \sqrt{b} :: 1$ vibr. : $\sqrt{\frac{b}{a}}$ the number of vibrations performed in 1 second; hence $1'' : t :: \sqrt{\frac{b}{a}} : t\sqrt{\frac{b}{a}} = \sqrt{\frac{b}{gf}}$, the constant number of vibrations.

So that the three numbers of vibrations, namely, of the ascending, constant, and descending pendulums, are proportional to the numbers 1·5708, 1, and hyp. log. $1 + \sqrt{2}$, or as 1·5708, 1, and ·88137; whatever be the length of the thread.

PROBLEM XX.

To determine the Circumstances of the Ascent and Descent of two unequal Weights, suspended at the two Ends of a Thread passing over a Pulley: the Weight of the Thread and of the Pulley being considered in the Solution.

Let l = the whole length of the thread;
a = the weight of the same;
b = Aw the dif. of lengths at first;
d = W − w the dif. of the two weights;
c = a weight applied to the circumference, such as to be equal to its whole wt. and friction reduced to the circumference;
s = W$+w+a+c$ the sum of the weights moved.

Then the weight of b is $\frac{ab}{l}$, and $d - \frac{ab}{l}$ is the moving force at first. But if x denote any variable space descended by W, or ascended by w, the difference of the lengths of the thread will be altered $2x$; so that the difference will then be $b-2x$, and its weight $\frac{b-2x}{l}a$; conseq. the motive force there will be $d - \frac{b-2x}{l}a = \frac{dl-ab+2ax}{l}$, and theref. $\frac{dl-ab+2ax}{sl} = f$ the accelerating force there. Hence then $v\dot{v} = 2gf\dot{x} = 2g\dot{x} \times \frac{dl-ab+2ax}{sl}$; the fluents of which give $v^2 = 4gx \times \frac{dl-ab+ax}{sl}$, or $v = 2\sqrt{\frac{ag}{sl}} \times \sqrt{(ex+x^2)}$ the general expression for the velocity, putting $e = \frac{dl-ab}{a}$. And when $x=b$, or W becomes as far below w as it was above it at the beginning, it is barely $v = 2\sqrt{\frac{bdg}{s}}$ for the velocity at that time. Also, when a, the weight of the thread, is nothing, the velocity is only $2\sqrt{\frac{dgx}{s}}$, as it ought.

Again, for the time, $\dot{t}$ or $\frac{\dot{x}}{v} = \frac{1}{2}\sqrt{\frac{sl}{ag}} \times \frac{\dot{x}}{\sqrt{(ex+x^2)}}$; the fluents of which give $t = \sqrt{\frac{sl}{ag}} \times \log. \frac{\sqrt{x} + \sqrt{(e+x)}}{\sqrt{e}}$ the general expression for the time of descending any space x.

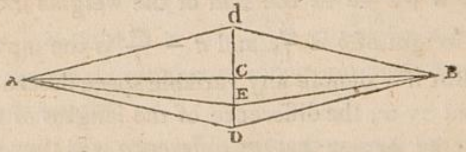
And if the radicals be expanded in a series, and the log. of it be taken, the same time will become

$$t = \sqrt{\frac{sx}{dg}} \times \sqrt{\frac{dl}{dl-ab}} \times \left(1 - \frac{x}{6e} + \frac{3x^2}{40e^2} \text{ \&c}\right).$$

Which therefore becomes barely $\sqrt{\frac{sx}{dg}}$ when a, the weight of the thread, is nothing, as it ought.

PROBLEM XXI.

To find the Velocity and Time of Vibration of a small Weight, fixed to the middle of a Line, or fine Thread void of Gravity, and stretched by a given Tension, the Extent of the Vibration being very small.



Let $l =$ AC half the length of the thread;
$a =$ CD the extent of the vibration;
$x =$ CE any variable distance from C;
$w =$ wt. of the small body fixed to the middle;
$\mathrm{w} =$ a wt. which, hung at each end of the thread, will be equal to the constant tension at each end, acting in the direction of the thread.

Now, by the nature of forces, AE : CE : : W the force in direction EA : the force in direction EC. Or, because AC is nearly = AE, the vibration being very small, taking AC instead of AE, it is AC : CE : W : $\frac{\mathrm{w}x}{l}$ the force in EC arising from the tension in EA. Which will be also the same for that in EB. Therefore the sum is $\frac{2\mathrm{w}x}{l}$ = the whole motive force in EC arising from the tensions on both sides. Consequently $\frac{2\mathrm{w}x}{lw} = f$ the accelerative force there. Hence the equation of the fluxions $v\dot{v}$ or $2gf\dot{s} = \frac{-4g\mathrm{w}x\dot{x}}{lw}$; and the flus. $v^2 = -\frac{4g\mathrm{w}x^2}{lw}$. But when $x = a$, this is $-\frac{4g\mathrm{w}a^2}{lw}$, and should be $= 0$; theref. the correct fluents are $v^2 = 4g\mathrm{w} \times \frac{a^2 - x^2}{lw}$, and hence $v = \sqrt{(4g\mathrm{w} \times \frac{a^2-x^2}{lw})}$ the velocity of the little body w at any point E. And when $x = 0$, it is $v = 2a\sqrt{\frac{g\mathrm{w}}{lw}}$ for the greatest velocity at the point C.

Now if we suppose $w = 1$ grain, $\text{W} = 5$lb troy, or 28800 grains, and $2l = \text{AB} = 3$ feet; the velocity at C becomes $a\sqrt{\frac{8 \times 16\frac{1}{12} \times 28800}{3}} = 1111\frac{2}{3}a$. So that

if $a = \frac{1}{10}$ inc. the greatest veloc. is $9\frac{5}{19}$ ft. per sec.
if $a = 1$ inc. the greatest veloc. is $92\frac{37}{60}$ ft. per sec.
if $a = 6$ inc. the greatest veloc. is $555\frac{7}{10}$ ft. per sec.

To find the time t, it is $\dot{t}$ or $\frac{-\dot{x}}{v} = \frac{1}{2}\sqrt{\frac{lw}{\text{W}g}} \times \frac{-\dot{x}}{\sqrt{(a^2 - x^2)}}$. Hence the correct fluent is $t = \frac{1}{2}\sqrt{\frac{wl}{\text{W}g}} \times$ arc to cosine $\frac{x}{a}$ and radius 1, for the time in DE. And when $x = 0$, the whole time in DC, or of half a vibration, is $\cdot 7854\sqrt{\frac{wl}{\text{W}g}}$; and conseq. the time of a whole vibration through Dd is $1\cdot 5708\sqrt{\frac{wl}{\text{W}g}}$.

Using the foregoing numbers, namely $w = 1$, $\text{W}=28800$, and $2l = 3$ feet; this expression for the time gives $\frac{1111\frac{2}{3}}{3\cdot 1416} = 353\frac{5}{9}$, the number of vibrations per second. But if $w = 2$, there would be 250 vibrations per second; and if $w = 100$, there would be $35\frac{16}{45}$ vibrations per second.

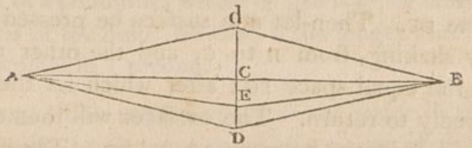
PROBLEM XXII.

To determine the same as in the last Problem, when the Distance CD *bears some sensible proportion to the Length* AB; *the Tension of the Thread however being still supposed a constant Quantity.*

Using here the same notation as in the last problem, and taking the true variable length AE for AC, it is AE or EB : CE :: $2\text{W} : \frac{2\text{W}x}{\text{AE}} = \frac{2\text{W}x}{\sqrt{(l^2 + x^2)}}$ the whole motive force from the two equal tensions W in AE and EB; and theref. $\frac{2\text{W}}{w} \times \frac{x}{\sqrt{(l^2+x^2)}} = f$ is the accelerative force at E. Theref. the fluxional equation is $v\dot{v}$ or $2gf\dot{s} = \frac{4\text{W}g}{w} \times \frac{-x\dot{x}}{\sqrt{(l^2+x^2)}}$; and the fluents $v^2 = \frac{8\text{W}g}{w} \times -\sqrt{(l^2 + x^2)}$. But when $x = a$, these are $0 = \frac{8\text{W}g}{w} \times -\sqrt{(l^2 + a^2)}$; therefore the correct fluents are $v^2 = \frac{8\text{W}g}{w} \times$

$[\sqrt{(l^2 + a^2)} - \sqrt{(l^2 + x^2)}] = \frac{8wg}{w} \times (\text{AD} - \text{AE})$. And hence $v = \sqrt{[\frac{8wg}{w} \times (\text{AD} - \text{AE})]}$ the general expression for the velocity at E. And when E arrives at C, it gives the greatest velocity there $= \sqrt{[\frac{8wg}{w} \times (\text{AD} - \text{AC})]}$. Which, when $\text{W} = 28800$, $w = 1$, $2l = 3$ feet, and $\text{CD} = 6$ inches or $\frac{1}{2}$ a foot, is $\sqrt{(8 \times 28800 \times 16\frac{1}{12} \times \frac{\sqrt{10} - 3}{2})} = 548\frac{1}{3}$ feet per second. Which came out $555\frac{7}{10}$ in the last problem, by using always AC for AE in the value of f. But when the extent of the vibrations is very small, as $\frac{1}{10}$ of an inch, as it commonly is, this greatest velocity here will be $\sqrt{8 \times 28800 \times 16\frac{1}{12} \times \frac{1}{43200}} = 9\frac{1}{4}$ nearly, which in the last problem was $9\frac{5}{19}$ nearly.

To find the time, it is $\dot{t}$ or $\frac{-\dot{x}}{v} = \sqrt{\frac{w}{8wg}} \times \frac{-\dot{x}}{\sqrt{[c - \sqrt{(l^2 + x^2)}]}}$, making $c = \text{AD} = \sqrt{(l^2 + a^2)}$. To find the fluent the easier, multiply the numer. and denom. both by $\sqrt{[c + \sqrt{(l^2 + x^2)}]}$, so shall $\dot{t} = \sqrt{\frac{w}{8wg}} \times \frac{-\dot{x}}{\sqrt{(a^2 - x^2)}} \times \sqrt{[c + \sqrt{(l^2 + x^2)}]}$. Expand now the quantity $\sqrt{[c + \sqrt{(l^2 + x^2)}]}$ in a series, and put $d = c + l$, so shall $\dot{t} = \sqrt{\frac{wd}{8wg}} \times \frac{-\dot{x}}{\sqrt{(a^2 - x^2)}}(1 + \frac{x^2}{4dl} - \frac{2d + l}{32d^2l^3}x^4 + \frac{4d^2 + 2dl + l^2}{128d^3l^5}x^6 - \frac{40d^3 + 8d^2l + 12dl^2 + 5l^3}{2048d^4l^7}x^8$ &c). Now the fluent of the first term $\frac{\dot{x}}{\sqrt{(a^2 - x^2)}}$ is $=$ the arc to sine $\frac{x}{a}$ and radius 1, which arc call A; and let P, Q, be the fluents of any other two successive terms, without the coefficients, the distance of Q from the first term A being n; then it is evident that $\dot{\text{Q}} = x^2\dot{\text{P}} = x^{2n}\dot{\text{A}}$, and $\dot{\text{P}} = x^{2n-2}\dot{\text{A}}$. Assume theref. $\text{Q} = b\text{P} - ex^{2n-1}\sqrt{(a^2 - x^2)}$; then is $\dot{\text{Q}}$ or $x^2\dot{\text{P}} = b\dot{\text{P}} - (2n - 1)ex^{2n-2}\dot{x}\sqrt{(a^2 - x^2)} + \frac{ex^{2n}\dot{x}}{\sqrt{(a^2 - x^2)}} = b\dot{\text{P}} - \frac{(2n - 1)ea^2x^{2n-2}\dot{x}}{\sqrt{(a^2 - x^2)}} + \frac{(2n - 1)ex^{2n}\dot{x}}{\sqrt{(a^2 - x^2)}} + \frac{ex^{2n}\dot{x}}{\sqrt{(a^2 - x^2)}} = b\dot{\text{P}} - (2n - 1)ea^2\dot{\text{P}} + (2n - 1)ex^2\dot{\text{P}} + ex^2\dot{\text{P}} = b\dot{\text{P}} - (2n - 1)ea^2\dot{\text{P}} + 2nex^2\dot{\text{P}}$. Then, comparing the coefficients of the like terms, we find $1 = 2en$, and $b = (2n - 1)ea^2$; from which are obtained $e = \frac{1}{2n}$, and $b = \frac{2n - 1}{2n}a^2$. Consequently $\text{Q} = \frac{(2n - 1)a^2\text{P} - x^{2n-1}\sqrt{(a^2 - x^2)}}{2n}$, the general equation between any two successive terms, and by means of which the series may be continued as far as we please. And hence, neglecting the coefficients, putting A = the first term, namely, the arc whose sine is $\frac{x}{a}$, and B, C, D, &c, the following terms, the series is as follows, $A + \frac{a^2 A - x\sqrt{(a^2 - x^2)}}{2} + \frac{3a^2 B - x^3\sqrt{(a^2 - x^2)}}{4} + \frac{5a^2 C - x^5\sqrt{(a^2 - x^2)}}{6}$ &c. Now when $x = 0$, this series $= 0$; and when $x = a$, the series becomes $\frac{1}{2}p + \frac{a^2 A}{2} + \frac{3a^2 B}{4} + \frac{5a^2 C}{6}$ &c, where $p = 3{\cdot}1416$, or the series is $\frac{1}{2}p(1 + \frac{1}{2}a^2 + \frac{1\,.\,3}{2\,.\,4}a^4 + \frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}a^6$ &c.)



So that, by taking in the coefficients, the general time of passing over any distance DE will be

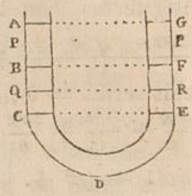
$\sqrt{\frac{w(c+l)}{8\mathrm{wg}}} \times \frac{1}{2}p \times (1 + \frac{1}{4dl} \cdot \frac{1}{2}a^2 - \frac{2d + l}{32d^2l^3} \cdot \frac{1\,.\,3}{2\,.\,4}a^4$ &c, $-$ arc sin. $\frac{x}{a} - \frac{1}{4dl} \cdot \frac{a^2 A - x\sqrt{(a^2 - x^2)}}{2} + \frac{2d + l}{32d^2l^3} \cdot \frac{3a^2 B - x^3\sqrt{(a^2 - x^2)}}{4}$ &c.

And hence, taking $x = 0$, and doubling, the time of a whole vibration, or double the time of passing over CD will be equal to $\frac{1}{2}p\sqrt{\frac{w(c+l)}{2\mathrm{wg}}} \times (1 + \frac{1}{4dl} \cdot \frac{1}{2}a^2 - \frac{2d + l}{32d^2l^3} \cdot \frac{1\,.\,3}{2\,.\,4}a^4 + \frac{4d^2 + 2dl + l^2}{128d^3l^5} \cdot \frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}a^6 - \frac{40d^3 + 8d^2l + 12dl^2 + 5l^3}{2048d^4l^7} \cdot \frac{1\,.\,3\,.\,5\,.\,7}{2\,.\,4\,.\,6\,.\,8}a^8$ &c.) Which, when $a = 0$, or $c = l$, becomes only $\frac{1}{2}p\sqrt{\frac{wl}{\mathrm{wg}}}$, the same as in the last problem, as it ought.

Taking here the same numbers as in the last problem, viz, $l = \frac{3}{2}$, $a = \frac{1}{2}$, $w = 2$, $\mathrm{w} = 28800$, $g = 16\frac{1}{12}$; then $\frac{1}{2}p\sqrt{\frac{w(c + l)}{2\mathrm{wg}}} = {\cdot}0040514$, and the series is $1 + {\cdot}006762 - {\cdot}000175 + {\cdot}000003$ &c $= 1{\cdot}006590$; therefore ${\cdot}0040514 \times 1{\cdot}006590 = {\cdot}0040965 = \frac{1}{245\frac{1}{5}}$ is the time of one whole vibration, and consequently $245\frac{1}{5}$ vibrations are performed in a second; which were 250 in the last problem.

PROBLEM XXIII.

It is proposed to determine the Velocity, and the Time of Vibration, of a Fluid in the Arms of a Canal or bent Tube.



Let the tube ABCDEF have its two branches AC, GE vertical, and the lower part CDE in any position whatever, the whole being of a uniform diameter or width throughout. Let water, or quicksilver, or any other fluid, be poured in, till it stand in equilibrio, at any horizontal line BF. Then let one surface be pressed or pushed down by shaking, from B to C, and the other will ascend through the equal space FG; after which let them be permitted freely to return. The surfaces will then continually vibrate in equal times between AC and EG. The velocity and times of which oscillations are therefore required.

When the surfaces are any where out of a horizontal line, as at P and Q, the parts of the fluid in QDR, on each side, below QR, will balance each other; and the weight of the part in PR, which is equal to 2PF, gives motion to the whole. So that the weight of the part 2PF is the motive force by which the whole fluid is urged, and therefore $\frac{\text{wt. of } 2\text{PF}}{\text{whole wt.}}$ is the accelerative force. Which weights being proportional to their lengths, if l be the length of the whole fluid, or axis of the tube filled, and $a =$ FG or BC; then is $\frac{a}{l}$ the accelerative force. Putting theref. $x =$ GP any variable distance, v the velocity, and t the time; then PF $= a - x$, and $\frac{2a-2x}{l} = f$ the accelerative force; hence $v\dot{v}$ or $2gf\dot{s} = \frac{4g}{l}(a\dot{x} - x\dot{x})$; the fluents of which give $v^2 = \frac{4g}{l}(2ax - x^2)$, and $v = \sqrt{(4g \times \frac{2ax - x^2}{l})}$ is the general expression for the velocity at any term. And when $x = a$, it becomes $v = 2a\sqrt{\frac{g}{l}}$ for the greatest velocity at B and F.

Again, for the time, we have $\dot{t}$ or $\frac{\dot{s}}{v} = \frac{1}{2}\sqrt{\frac{l}{g}} \times \frac{\dot{x}}{\sqrt{(2ax - x^2)}}$; the fluents of which give $t = \frac{1}{2}\sqrt{\frac{l}{g}} \times$ arc to versed sine $\frac{x}{a}$ and radius 1, the general expression for the time. And when $x = a$, it becomes $t = \frac{1}{4}p\sqrt{\frac{l}{g}}$ for the time of moving from G to F, p being $= 3{\cdot}1416$; and consequently $\frac{1}{2}p\sqrt{\frac{l}{g}}$ the time of a whole vibration from G to E, or from C to A. And which therefore is the same, whatever AB is, the whole length l remaining the same.

And the time of vibration is also equal to the time of the vibration of a pendulum whose length is $\frac{1}{2}l$, or half the length of the axis of the fluid. So that, if the length l be $78\frac{1}{4}$ inches, it will oscillate in 1 second.

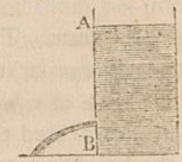
Scholium. This reciprocation of the water in the canal, is nearly similar to the motion of the waves of the sea. For the time of vibration is the same, however short the branches are, provided the whole length be the same. So that when the height is small, in proportion to the length of the canal, the motion is similar to that of a wave, from the top to the bottom or hollow, and from the bottom to the top of the next wave; being equal to two vibrations of the canal; the whole length of a wave, from top to top, being double the length of the canal. Hence the wave will move forward by a space nearly equal to its breadth, in the time of two vibrations of a pendulum whose length is ($\frac{1}{2}l$) half the length of the canal, or one-fourth of the breadth of a wave, or in the time of one vibration of a pendulum whose length is the whole breadth of the wave, since the times of vibration are as the square roots of their lengths. Consequently, waves whose breadth is equal to $39\frac{1}{8}$ inches, or $3\frac{25}{96}$ feet, will move over $3\frac{25}{96}$ feet in a second, or $195\frac{5}{8}$ feet in a minute, or nearly 2 miles and a quarter in an hour. And the velocity of greater or less waves will be increased or diminished in the subduplicate ratio of their breadths.

Thus, for instance, for a wave of 18 inches breadth, as $\sqrt{39\frac{1}{8}} : 39\frac{1}{8} :: \sqrt{18} : \sqrt{(39\frac{1}{8} \times 18)} = \frac{3}{2}\sqrt{313} = 26{\cdot}5377$ the velocity of the wave of 18 inches breadth.

PROBLEM XXIV.

To assign the Velocity with which Water, or other Fluids, spouts out from the bottom of a Vessel.

The velocity with which water runs out by a hole in the bottom or side of a vessel, is equal to that which is generated by gravity through the height of the fluid above the hole; that is, the velocity of a heavy body acquired by falling freely through the height AB.



For, divide the altitude AB into a great number of very small parts, each being 1, their number a, or $a =$ the altitude AB.

Now, by prop. 61, vol. 2 of the Course, the pressure of the fluid against the hole B, by which the motion is generated, is equal to the weight of the column of fluid above it, that is the column whose height is AB or a, and base the area of the hole B. Therefore the pressure on the hole, or small part of the fluid 1, is to its weight, or the natural force of gravity, as a to 1. But the velocities generated in the same body in any time, are as those forces; and because gravity generates the velocity 2 in descending through the small space 1, therefore $1 : a :: 2 : 2a$, the velocity generated by the pressure of the column of fluid in the same time. But $2a$ is also the velocity generated by gravity in descending through a or AB. That is, the velocity of the issuing water, is equal to that which is acquired by a body in falling through the height AB.

The same otherwise.

Because the momenta, or quantities of motion, generated in two given bodies, by the same force, acting during the same or an equal time, are equal. And as the force in this case, is the weight of the superincumbent column of the fluid over the hole. Let the one body to be moved, be that column itself, expressed by ah, where a denotes the altitude AB, and h the area of the hole; and the other body is the column of the fluid that runs out uniformly in one second suppose, with the middle or medium velocity of that interval of time, which is $\frac{1}{2}hv$, if v be the whole velocity required. Then the mass $\frac{1}{2}hv$, with the velocity v, gives the quantity of motion $\frac{1}{2}hv \times v$, or $\frac{1}{2}hv^2$, generated in 1 second, in the spouting water: also $2g$ or $32\frac{1}{6}$ feet, is the velocity generated in the mass ah, during the same interval of one second; consequently $ah \times 2g$, or $2ahg$, is the motion generated in the column ah in the same time of one second. But as these two momenta must be equal, this gives $\frac{1}{2}hv^2 = 2ahg$: hence then $v^2 = 4ag$, and $v = 2\sqrt{ag}$, for the value of the velocity sought; which therefore is exactly the same as the velocity generated by the gravity in falling through the space a, or the whole height of the fluid.

For example, if the fluid were air, of the whole height of the atmosphere, supposed uniform, which is about $5\frac{1}{4}$ miles, or 27720 feet $= a$. Then $2\sqrt{ag} = 2\sqrt{27720 \times 16\frac{1}{12}} = 1335$ feet $= v$ the velocity, that is, the velocity with which common air would rush into a vacuum.

Corol. 1. The velocity, and quantity run out, at different depths, are as the square roots of the depths. For the velocity acquired in falling through AB, is as $\sqrt{AB}$.

Corol. 2. The fluid spouts out with the same velocity, whether it be downward or upward, or sideways; because the pressure of fluids is the same in all directions, at the same depth. And therefore, if an adjutage be turned upward, the jet will ascend to the height of the surface of the water in the vessel. And this is confirmed by experience, by which it is found that jets really ascend nearly to the height of the reservoir, abating a small quantity only, for the friction against the sides, and some resistance from the air and from the oblique motion of the fluid in the hole.

Corol. 3. The quantity run out in any time, is equal to a column or prism, whose base is the area of the hole, and its length the space described in that time by the velocity acquired by falling through the altitude of the fluid. And the quantity is the same, whatever be the figure of the orifice, if it is of the same area.

Therefore, if a denote the altitude of the fluid,
and h the area of the orifice,
also $g = 16\frac{1}{12}$ feet, or 193 inches;
then $2h\sqrt{ag}$ will be the quantity of water discharged in a second of time; or nearly $8\frac{1}{45}h\sqrt{a}$ cubic feet, when a and h are taken in feet.

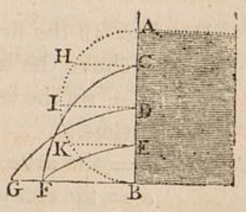
So, for example, if the height a be 25 inches, and the orifice $h = 1$ square inch; then $2h\sqrt{ag} = 2\sqrt{25 \times 193} =$ 139 cubic inches, which is the quantity that would be discharged per second.

SCHOLIUM.

When the orifice is in the side of the vessel, then the velocity is different in the different parts of the hole, being less in the upper parts of it than in the lower. However, when the hole is but small, the difference is inconsiderable, and the altitude may be estimated from the centre of the hole, to obtain the mean velocity. But when the orifice is pretty large, then the mean velocity is to be more accurately computed by other principles, given in the next problem.

It is not to be expected that experiments, on the quantity of water run out, will exactly agree with this theory, as well on account of the resistance of the air, as the resistance of the water against the sides of the orifice, and the oblique motion of the particles of the water on entering it. For, it is not merely the particles situated immediately in the column over the hole, which enter it and issue forth, as if that column only were in motion; but also particles from all the surrounding parts of the fluid, which is in commotion quite around; and the particles thus entering the hole in all directions, strike against each other, and impede one another's motion: from which it happens, that it is the particles in the centre of the hole only that issue out with the whole velocity due to the entire height of the fluid, while the other particles towards the sides of the orifice pass out with decreased velocities; and hence the medium velocity through the orifice, is somewhat less than that of a single body only, urged with the same pressure of the superincumbent column of the fluid. And experiments, on the quantity of water discharged through apertures, show that the quantity must be diminished, by those causes, rather more than the fourth part, when the orifice is small, or such as to make the mean velocity nearly equal to that in a body falling through half the height of the fluid above the orifice.

Experiments have also been made on the extent to which the spout of water ranges on a horizontal plane, and compared with the theory, by calculating it as a projectile discharged with the velocity acquired by descending through the height of the fluid. For, when the aperture is in the side of the vessel, the fluid spouts out horizontally with a uniform velocity, which, combined with the perpendicular velocity from the action of gravity, causes the jet to form the curve of a parabola. Then the distances to which the jet spouts on the horizontal plane BG, are as the roots of the rectangles of the segments AC . CB, AD . DB, AE . EB. For the ranges BF, BG, are as the times and horizontal velocities; but the velocity is as $\sqrt{AC}$; and the time of the fall, which is the same as the time of moving, is as $\sqrt{CB}$; therefore the distance BF is as $\sqrt{(AC . CB)}$; and the distance BG as $\sqrt{(AD . DB)}$. And hence, if two holes be made equidistant from the top and bottom, they will project the water to the same distance; for if AC = EB, then the rectangle



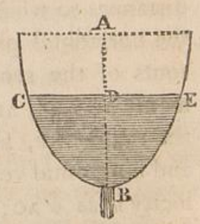
AC . CB is equal the rectangle AE . EB: which makes BF the same for both. Or, if on the diameter AB a semicircle be described; then, because the squares of the ordinates CH, DI, EK are equal to the rectangles AC . CB, &c; therefore the distances BF, BG are as the ordinates CH, DI. And hence also it follows that the projection from the middle point D will be farthest, for DI is the greatest ordinate.

These are the proportions of the distances: but for the absolute distances, it will be thus. The velocity through any hole C, is such as will carry the water horizontally through a space equal to 2AC in the time of falling through AC: but, after quitting the hole, it describes a parabola, and comes to F in the time a body will fall through CB; and to find this distance, since the times are as the roots of the spaces, therefore $\sqrt{AC} : \sqrt{CB} :: 2AC : 2\sqrt{(AC \cdot CB)} = 2CH = BF$, the space ranged on the horizontal plane. And the greatest range $BG = 2DI$, or $2AD$, or equal to AB.

And as these ranges answer very exactly to the experiments, this confirms the theory, as to the velocity assigned.

PROBLEM XXIV.

To assign the Time of emptying a Vessel of Water, or other Fluid, through a Hole in the Bottom, as ACBE.



Put h = AB the first height of the fluid above the hole;

x = DB the variable altitude at any time;

a = the area of the orifice B;

a' = the area of the descending surface CE.

Now the velocity of any issuing uniform fluid, being the same as that acquired by a body in falling through the height DB of the fluid, which is as the square-root of the height; and 32 being the velocity acquired in falling through the space

16 feet; therefore $\sqrt{16} : 32 :: \sqrt{DB}$ or $\sqrt{x} : 8\sqrt{x}$ the velocity of falling through DB, or of the issuing fluid at B; and $8a\sqrt{x}$ the quantity issuing in 1 second; or rather, reducing this by $\frac{1}{4}$, then $4 : 3 :: 8a\sqrt{x} : 6a\sqrt{x}$ for the quantity issuing per second, allowing $\frac{1}{4}$ for the contraction of the stream. But the content of the space descended by the surface CE, is always equal to what issues by the orifice at B; therefore the velocities at B and of the descending section CE, are always reciprocally proportional, that is, $a' : a :: 6\sqrt{x} : \frac{6a\sqrt{x}}{a'}$ the velocity, per second, of the descending section. Now, in descending the space $\dot{x}$, the velocity may be considered as uniform; and uniform descents, or uniformly described spaces, are as their times; therefore $\frac{6a\sqrt{x}}{a'} : \dot{x} :: 1'' : \frac{a'\dot{x}}{6a\sqrt{x}} = \dot{t}$, the time of descending $\dot{x}$ space, or the fluxion of the time of exhausting.

As to the fluent of this fluxion, it will be various, according to the figure of the vessel. If this be a prism of any kind whatever, the quantity a', or the section CE, will be always the same constant quantity; and then the fluent will be $\frac{a'}{3a}\sqrt{x} = t$ the time of exhausting the prism, whose base is a', and altitude x.

But, for any other figure, the quantity a', or CE, will be a variable quantity, and will alter the form of the fluent, according to the nature of the figure. So, if the vessel be a paraboloid for example, the nature of which is such, that the different sections are proportional to their distances from the vertex of the curve; that is, as BA : b the section at A :: BD : CE, or as $h : b :: x : \frac{bx}{h} = a'$, the value of a'; which being substituted for it in the general value of $\dot{t}$, it becomes $\dot{t} = \frac{b\sqrt{x}}{6ah}\dot{x}$; the whole fluent of which is $\frac{b\sqrt{h}}{9a}$, for the time of the complete exhausting of the vessel ACBE at the vertex downwards.

But if the vertex were upwards, the fluxion of the time would be $\dot{t} = \frac{h-x}{\sqrt{x}} \times \frac{b\dot{x}}{6ah}$, the whole fluent of which would be $t = \frac{2b\sqrt{h}}{9a}$, for the time of exhausting the paraboloid at the base, or the vertex upwards. Hence it may be observed, that this latter time is just double the former, when the vertex was downwards; also that this last, with the vertex downwards, is but the $\frac{1}{3}$ part of the time for the prism of equal base and altitude first above found; or that the three times are in proportion respectively as the three numbers 3, 2, 1, having all the same base and altitude; also that all their times are proportional to the base and root of the altitude directly, and as the orifice inversely.

The times for many other figures may be seen in the first article in my Mathematical Miscellany.

In the above investigation, the resistance made by the air to the issuing fluid, is neglected, the issue being supposed to be made in a vacuum. But the difference of effect ought not to be neglected, when the density of the medium bears any considerable proportion to that of the issuing fluid. It will make no difference in the account, in whatever part of the base the aperture is placed, the altitude of the surface above it being the only consideration. Nor is it material what the figure of it is; whether circular, square, triangular, &c, or regular or not; its area alone being the only necessary consideration.

PROBLEM XXV.

To determine the Issue of a Fluid through a Notch at the Top of a Vessel.

If a notch or slit EH in form of a parallelogram, be cut in the side of a vessel, full of water, AD; the quantity of water flowing through it, will be $\frac{2}{3}$ of the quantity flowing through an equal orifice, placed at the whole depth EG, or at the base GH, in the same time; it being supposed that the vessel is always kept full.

For, the velocity at GH is to the velocity at IL, as $\sqrt{EG}$ to $\sqrt{EI}$; that is, as GH or IL to IK, the ordinate of a parabola EKH, whose axis is EG. Therefore the sum of the velocities at all the points I, is to as many times the velocity at G, as the sum of all the ordinates IK, to the sum of all the IL's; namely, as the area of the parabola EGH, is to the area EGHF; that is, the quantity running through the notch EH, is to the quantity running through an equal horizontal area placed at GH, as EGHKE, to EGHF, or as 2 to 3; the area of a parabola being $\frac{2}{3}$ of its circumscribing parallelogram.

Corol. 1. The mean velocity of the water in the notch, is equal to $\frac{2}{3}$ of that at GH.

Corol. 2. The quantity flowing through the hole IGHL, is to that which would flow through an equal orifice placed as low as GH, as the parabolic frustum IGHK, is to the rectangle IGHL. As appears from the demonstration.

PROBLEM XXVI.

To determine the Time of filling the Ditches of a Work with Water at the Top, by a Sluice of 2 *Feet square; the Head of Water above the Sluice being* 10 *Feet, and the Dimensions of the Ditch being* 20 *Feet wide at Bottom,* 22 *at Top,* 9 *deep, and* 1000 *Feet long.*

The capacity of the ditch is 189000 cubic feet.

But $\sqrt{16} : \sqrt{10} :: 32 : 8\sqrt{10}$ the velocity of the water through the sluice, the area of which is 4 square feet; therefore $32\sqrt{10}$ is the quantity per second running through it; or $24\sqrt{10}$ only when reduced $\frac{1}{4}$ for the contraction of the stream; consequently $24\sqrt{10} : 189000 :: 1'' : \frac{7875}{\sqrt{10}} = 2490''$ or $41'\ 10''$ nearly, is the time of filling the ditch.

PROBLEM XXVII.

To determine the Time of emptying a Vessel of Water by a Sluice in the Bottom of it, or in the Side near the Bottom, the Height of the Aperture being very small in respect of the Altitude of the Fluid.

Put $a =$ the area of the aperture or sluice;
$d =$ the whole depth of water;
$x =$ the variable alt. of the surface above the aperture;
$a' =$ the area of the surface of the water.

Then $\sqrt{16} : \sqrt{x} :: 32 : 8\sqrt{x}$ the velocity with which the fluid will issue at the sluice; and hence $a' : a :: 8\sqrt{x} : \frac{8a\sqrt{x}}{a'}$ the velocity with which the surface of the water will descend at the altitude x, or the space it would descend in 1 second with the velocity there, or it is only $\frac{6a\sqrt{x}}{a'}$ when reduced $\frac{1}{4}$ for the contraction of the stream. Now in descending the space $\dot{x}$, the velocity may be considered as uniform; and uniform descents are as their times; therefore $\frac{6a\sqrt{x}}{a'} : \dot{x} :: 1'' : \frac{a'\dot{x}}{6a\sqrt{x}}$ the time of descending $\dot{x}$ space, or the fluxion of the time of exhausting. That is, $\dot{t} = \frac{-a'\dot{x}}{6a\sqrt{x}}$.

Now, when the nature or figure of the vessel is given, there will be given a' in terms of x; which value of a' being substituted into this fluxion of the time, the fluent of the result will be the time of exhausting sought.

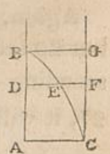
So if, for example, the vessel be any prism, or every where of the same breadth; then a' is a constant quantity, and therefore the fluent is $-\frac{a'}{3a}\sqrt{x}$. But when $x = d$, this becomes $-\frac{a'}{3a}\sqrt{d}$, and should be 0; therefore the correct fluent is $t = \frac{a'}{3a} \times (\sqrt{d} - \sqrt{x})$ for the time of the surface descending till the depth of the water be x. And when $x = 0$, the whole time of exhausting is barely $\frac{a'}{3a}\sqrt{d}$.

And hence if a' be 10000 square feet, $a = 1$ square foot, and $d = 10$ feet; the time is 10540 seconds, or $2^h\ 55'\ 40''$.

Again, if the vessel be a ditch, or canal, of 20 feet broad at the bottom, 22 at the top, 9 deep, and 1000 feet long; then is $90 : 90 + x :: 20 : \frac{90 + x}{90} \times 20$ the breadth of the surface of the water when its depth in the canal is x; and therefore $a = \frac{90 + x}{90} \times 20000$ is the surface at that time. Consequently $\dot{t}$ or $\frac{-a'\dot{x}}{6a\sqrt{x}} = 10000 \times \frac{90 + x}{90} \times \frac{-\dot{x}}{3a\sqrt{x}}$ is the fluxion of the time; the correct fluent of which, when $x = 0$, is $\frac{10000}{3} \times \frac{180 + \frac{2}{3}d}{90a} \times \sqrt{d} = \frac{10000 \times 186 \times 3}{90 \times 3} = 20666''$ nearly, or $5^h\ 44'\ 26''$, the whole time of exhausting by a sluice of 1 foot square.

PROBLEM XXVIII.

To determine the Time of emptying any Ditch, or Inundation, &c, by a Cut or Notch, from the Top to the Bottom of it.



Let x = AB, the variable height of the descending water at any time;
b = AC, the breadth of the cut;
d = the whole or first depth of water;
A = the area of the surface of the water in the ditch;
$g = 16\frac{1}{12}$ feet, the descent by gravity in $1''$.

Now, the velocity at any point D, is as $\sqrt{\text{BD}}$, that is as the ordinate DE of a parabola BEC, whose base is AC, and altitude AB. Therefore the velocities at all the points in AB, are as all the ordinates of the parabola. Consequently, the quantity of water running through the cut ABGC, in any time, is to the quantity which would run through an equal aperture placed all at the bottom, in the same time, as the area of the parabola ABC, to the area of the parallelogram ABGC, that is, as 2 to 3.

But $\sqrt{g} : \sqrt{x} :: 2g : 2\sqrt{gx}$, the velocity at AC; therefore $2\sqrt{gx} \times bx \times \frac{2}{3} = \frac{4}{3}bx\sqrt{gx}$ is the quantity discharged per second through ABGC; and consequently $\frac{4bx\sqrt{gx}}{3\text{A}}$ is the velocity per second of the descending surface. Hence then

$\frac{4bx\sqrt{gx}}{3A} : -\dot{x} :: 1'' : \frac{-3A\dot{x}}{4bx\sqrt{gx}} = \dot{t}$, the fluxion of the time of descending.

Now when A the surface of the water is constant, or the ditch is equally broad throughout, the correct fluent of this fluxion gives $t = \frac{3A}{2b\sqrt{g}} \times \frac{\sqrt{d}-\sqrt{x}}{\sqrt{dx}}$ for the general time of sinking the surface to any depth x. And when $x = 0$, this expression is infinite; which shows that the time of a complete exhaustion is infinite.

But if $d = 9$ feet, $b = 2$ feet, $A = 21 \times 1000 = 21000$, and it be required to exhaust the water down to $\frac{1}{16}$ of a foot deep; then $x = \frac{1}{16}$, and the above expression becomes $\frac{3 \times 21000}{4 \times 4\frac{1}{24}} \times \frac{3-\frac{1}{4}}{\frac{3}{4}} = 14400''$, or just 4 hours for that time. And if it be required to depress it 8 feet, or till 1 foot depth of water remain in the ditch, the time of sinking the water to that point will be 43′ 38″.

Again, if the ditch be the same depth and length as before, but 20 feet broad at bottom, and 22 at top; then the descending surface will be a variable quantity, and, by prob. 27, it will be $\frac{90+x}{90} \times 20000$; hence in this case the flux. of the time, or $\frac{-3A\dot{x}}{4bx\sqrt{gx}}$, becomes $\frac{-500}{3b\sqrt{g}} \times \frac{90+x}{x\sqrt{x}}\dot{x}$; the correct fluent of which is $t = \frac{1000}{3b\sqrt{g}} \times (\frac{90-x}{\sqrt{x}} - \frac{90-d}{\sqrt{d}})$ for the time of sinking the water to any depth x.

Now when $x = 0$, this expression for the complete exhaustion becomes infinite.

But if . . $x = 1$ foot, the time t is 42′ 56″$\frac{1}{2}$.

And when $x = \frac{1}{16}$ foot, the time is 3^h 50′ 28″ $\frac{1}{2}$.

PROBLEM XXIX.

To determine the Time of filling the Ditches of a Fortification 6 Feet deep with Water, through the Sluice of a Trunk of 3 Feet Square, the Bottom of which is level with the Bottom of the Ditch, and the Height of the supplying Water is 9 Feet above the Bottom of the Ditch.

Let ACDB represent the area of the vertical sluice, being a square of 9 square feet, and AB level with the bottom of the ditch. And suppose the ditch filled to any height AE, the surface being then at EF.

Put $a = 9$ the height of the head or supply;
$b = 3 = \text{AB} = \text{AC}$;
$g = 16\frac{1}{12}$;
$\text{A} =$ the area of a horizontal section of the ditches;
$x = a - \text{AE}$, the height of the head above EF.

Then $\sqrt{g} : \sqrt{x} :: 2g : 2\sqrt{gx}$ the velocity with which the water presses through the part AEFB; and theref. $2\sqrt{gx} \times \text{AEFB} = 2b\sqrt{gx}\,(a - x)$ is the quantity per second running through AEFB. Also, the quantity running per second through ECDF is $2\sqrt{gx} \times \frac{11}{12}\text{ECDF} = \frac{11}{6}b\sqrt{gx}(b - a + x)$ nearly. For the real quantity is, by proceeding as in the last prob. the difference between two parab. segs. the alt. of the one being x, its base b, and the alt. of the other $a - b$; and the medium of that dif. between its greatest state at AB, where it is $\frac{9}{10}\text{AD}$, and its least state at CD, where it is 0, is nearly $\frac{11}{12}\text{ED}$. Consequently the sum of the two, or $\frac{1}{6}b\sqrt{gx}\,(a + 11b - x)$ is the quantity per second running in by the whole sluice ACDB. Hence then $\frac{1}{6}b\sqrt{gx} \times \frac{a + 11b - x}{\text{A}} = v$ is the rate or velocity per second with which the water rises in the ditches; and so $v : -\dot{x} :: 1'' : \dot{t} = -\frac{\dot{x}}{v} = \frac{-6\text{A}}{b\sqrt{g}} \times \frac{x^{-\frac{1}{2}}\dot{x}}{c - x}$ the fluxion of the time of filling to any height AE, putting $c = a + 11b$.

Now when the ditches are of equal width throughout, A is a constant quantity, and in that case a correct fluent of this fluxion is $t = \frac{6\text{A}}{b\sqrt{gc}} \times \log.\,(\frac{\sqrt{c} + \sqrt{a}}{\sqrt{c} - \sqrt{a}} \times \frac{\sqrt{c} - \sqrt{x}}{\sqrt{c} + \sqrt{x}}$ the general expression for the time of filling to any height AE, or $a - x$, not exceeding the height AC of the sluice. And when $x = \text{AC} = a - b = d$ suppose, then $t = \frac{6\text{A}}{b\sqrt{gc}} \times \log.$

$(\frac{\sqrt{c}+\sqrt{a}}{\sqrt{c}-\sqrt{a}} \cdot \frac{\sqrt{c}-\sqrt{d}}{\sqrt{c}+\sqrt{d}})$ is the time of filling to CD the top of the sluice.

Again, for filling to any height GH above the sluice, x denoting as before $a -$ AG the height of the head above GH, $2\sqrt{gx}$ will be the velocity of the water through the whole sluice AD: and therefore $2b^2\sqrt{gx}$ the quantity per second, and $\frac{2b^2\sqrt{gx}}{A} = v$ the rise per second of the water in the ditches; consequently $v : -\dot{x} :: 1'' : \dot{t} = -\frac{\dot{x}}{v} = \frac{-A}{2b^2\sqrt{g}} \times \frac{\dot{x}}{\sqrt{x}}$ the general fluxion of the time; the correct fluent of which, being 0 when $x = a - b = d$, is $t = \frac{A}{b^2\sqrt{g}}(\sqrt{d} - \sqrt{x})$ the time of filling from CD to GH.

Then the sum of the two times, namely, that of filling from AB to CD, and that of filling from CD to GH, is $\frac{A}{b\sqrt{g}}[\frac{\sqrt{d}-\sqrt{x}}{b} + \frac{6}{\sqrt{x}} \log. (\frac{\sqrt{c}+\sqrt{a}}{\sqrt{c}-\sqrt{a}} \cdot \frac{\sqrt{c}-\sqrt{d}}{\sqrt{c}+\sqrt{d}})]$ for the whole time required. And, using the numbers in the prob., this becomes $\frac{A}{3\sqrt{g}}[\frac{\sqrt{6}-\sqrt{3}}{3} + \frac{6}{\sqrt{42}} \times 1.(\frac{\sqrt{42}+\sqrt{9}}{\sqrt{42}-\sqrt{9}} \cdot \frac{\sqrt{42}-\sqrt{6}}{\sqrt{42}+\sqrt{6}})] =$ 0·03577277A, the time in terms of A the area of the length and breadth, or horizontal section of the ditches. And if we suppose that area to be 200000 square feet, the time required will be $7154''$, or $1^{h}\ 59'\ 14''$.

And if the sides of the ditch slope a little, so as to be a little narrower at the bottom than at top, the process will be nearly the same, substituting for A its variable value, as in the preceding problem. And the time of filling will be very nearly the same as that above determined.

PROBLEM XXX.

But if the Water, from which the Ditches are to be filled, be the Tide, which at Low Water is below the Bottom of the Trunk, and rises to suppose 9 Feet above the Bottom of it by a regular Rise of One Foot in Half an Hour; it is required to *ascertain the Time of Filling it to 6 Feet high, as before in the last Problem.*

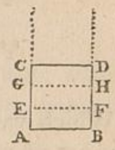
Let ACDB represent the sluice; and when the tide has risen to any height GH, below CD the top of the sluice, without the ditches, let FF be the mean height of the water within. And put $b = 3 = \text{AB} = \text{AC}$;

$g = 16\frac{1}{12}$;

$\text{A} =$ horizontal section of the ditches;

$x = \text{AG}$;

$z = \text{AE}$.



Then $\sqrt{g} : \sqrt{\text{EG}} :: 2g : 2\sqrt{g(x-z)}$ the velocity of the water through AEFB; and

$\sqrt{g} : \sqrt{\text{EG}} :: \frac{4}{3}g : \frac{4}{3}\sqrt{g(x-z)}$ the mean vel. through EGHF;

theref. $2bz\sqrt{g(x-z)}$ is the quantity per sec. through AEFB;

and $\frac{4}{3}b(x-z)\sqrt{g(x-z)}$ is the same through EGHF;

conseq. $\frac{2}{3}b\sqrt{g} \times (2x+z)\sqrt{(x-z)}$ is the whole through AGHB per second. This quantity divided by the surface A, gives $\frac{2b\sqrt{g}}{3\text{A}} \times (2x+z)\sqrt{(x-z)} = v$ the velocity per second with which EF, or the surface of the water in the ditches, rises. Therefore

$$v : \dot{z} :: 1'' : \dot{t} = \frac{\dot{z}}{v} = \frac{3\text{A}}{2b\sqrt{g}} \times \frac{\dot{z}}{(2x+z)\sqrt{(x-z)}}.$$

But as GH rises uniformly 1 foot in $30'$ or $1800''$, therefore $1 : \text{AG} :: 1800'' : 1800x = t$ the time of the tide rising through AG; conseq. $\dot{t} = 1800\dot{x} = \frac{3\text{A}}{2b\sqrt{g}} \times \frac{\dot{z}}{(2x+z)\sqrt{(x-z)}}$, or $m\dot{z} = (2x+z)\sqrt{(x-z)} \cdot \dot{x}$ is the fluxional equa. expressing the relation between x and z; where $m = \frac{\text{A}}{1200b\sqrt{g}} = \frac{3200}{231}$ or $13\frac{197}{231}$ when $\text{A} = 200000$ square feet.

Now to find the fluent of this equation, assume $z = \text{A}x^{\frac{5}{2}} + \text{B}x^{\frac{8}{2}} + \text{C}x^{\frac{11}{2}} + \text{D}x^{\frac{14}{2}}$ &c. So shall

$$\sqrt{(x-z)} = x^{\frac{1}{2}} - \frac{\text{A}}{2}x^{\frac{4}{2}} - \frac{\text{A}^2 + 4\text{B}}{8}x^{\frac{7}{2}} - \frac{\text{A}^3 + 4\text{AB} + 8\text{C}}{16}x^{\frac{10}{2}} \text{ \&c,}$$

$$2x + z = 2x + \text{A}x^{\frac{5}{2}} + \text{B}x^{\frac{8}{2}} + \text{C}x^{\frac{11}{2}} \text{ \&c,}$$

$$(2x+z)\sqrt{(x-z)}\dot{x} = 2x^{\frac{3}{2}}\dot{x} * - \frac{3\text{A}^2}{4}x^{\frac{9}{2}}\dot{x} - \frac{\text{A}^3 + 6\text{AB}}{4}x^{\frac{12}{2}}\dot{x} \text{ \&c,}$$

and $m\dot{z} = \frac{5}{2}m\text{A}x^{\frac{3}{2}}\dot{x} + \frac{8}{2}m\text{B}x^{\frac{6}{2}}\dot{x} + \frac{11}{2}m\text{C}x^{\frac{9}{2}}\dot{x} + \frac{14}{2}m\text{D}x^{\frac{12}{2}}\dot{x}$ &c.

Then equating the coefficients of the like terms,

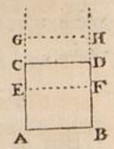
so shall	and consequently
$\frac{5}{2}m\text{A} = 2,$	$\text{A} = \frac{4}{5m},$
$\frac{8}{2}m\text{B} = 0,$	$\text{B} = 0,$
$\frac{11}{2}m\text{C} = -\frac{3}{4}\text{A}^2,$	$\text{C} = -\frac{24}{275m^3},$
$\frac{14}{2}m\text{D} = -\frac{3}{4}\text{A}^3 - \frac{3}{2}\text{AB},$	$\text{D} = -\frac{16}{875m^4},$
&c ;	&c.

Which values of A, B, C, &c, substituted in the assumed value of z, give

$$z = \frac{4}{5m}x^{\frac{5}{2}} * - \frac{24}{275m^3}x^{\frac{11}{2}} - \frac{16}{875m^4}x^{\frac{14}{2}} \text{ \&c ;}$$

$$\text{or } z = \frac{4}{5m}x^{\frac{5}{2}} \text{ very nearly.}$$

And when $x = 3 = \text{AC}$, then $z = {\cdot}886$ of a foot, or $10\frac{2}{3}$ inches, $= \text{AE}$, the height of the water in the ditches when the tide is at CD or 3 feet high without, or in the first hour and half of time.



Again, to find the time, after the above, when EF arrives at CD, or when the water in the ditches arrives as high as the top of the sluice.

The notation remaining as before,

then $2bz\sqrt{g(x-z)}$ per sec. runs through AF,

and $\frac{8}{5}b(3-z)\sqrt{g(x-z)}$ per sec. thro' ED nearly;

therefore $\frac{2}{5}b\sqrt{g} \times (12+z)\sqrt{(x-z)}$ is the whole per second through AD nearly.

conseq. $\frac{2b\sqrt{g}}{5\text{A}} \times (12+z)\sqrt{(x-z)} = v$ is the velocity per second of the point E ; and therefore

$$v : \dot{z} :: 1'' : \dot{t} = \frac{\dot{z}}{v} = \frac{5\text{A}}{2b\sqrt{g}} \times \frac{\dot{z}}{(12+z)\sqrt{(x-z)}} = 1800\dot{x}, \text{ or}$$

$$m\dot{z} = (12+z)\sqrt{(x-z)}\,.\,\dot{x}, \text{ where } m = \frac{\text{A}}{720b\sqrt{g}} = 23\tfrac{2}{23} \text{ nearly.}$$

Assume $z = \text{A}x^{\frac{3}{2}} + \text{B}x^{\frac{4}{2}} + \text{C}x^{\frac{5}{2}} + \text{D}x^{\frac{6}{2}}$ &c. So shall

$$\sqrt{(x-z)} = x^{\frac{1}{2}} - \frac{\text{A}}{2}x^{\frac{2}{2}} - \frac{\text{A}^2+4\text{B}}{8}x^{\frac{3}{2}} - \frac{\text{A}^3+4\text{AB}+8\text{C}}{16}x^{\frac{4}{2}} \text{ \&c ;}$$

$$12 + z = 12 + \text{A}x^{\frac{3}{2}} + \text{B}x^{\frac{4}{2}} + \text{C}x^{\frac{5}{2}} \text{ \&c ;}$$

$(12+z) \cdot \sqrt{(x-z)} \cdot \dot{x} = 12x^{\frac{1}{2}}\dot{x} - 6\text{A}x^{\frac{2}{2}}\dot{x} - (\frac{3}{2}\text{A}^2 + 6\text{B})x^{\frac{3}{2}}\dot{x}$ &c;

$m\dot{z} = \frac{3}{2}m\text{A}x^{\frac{1}{2}}\dot{x} + \frac{4}{2}m\text{B}x^{\frac{2}{2}}\dot{x} + \frac{5}{2}m\text{C}x^{\frac{3}{2}}\dot{x}$ &c.

Then, equating the like terms, &c, we have

$\text{A} = \frac{8}{m}$, $\text{B} = -\frac{24}{m^2}$, $\text{C} = \frac{96}{5m^3}$, $\text{D} = \frac{64}{3m^4}$ nearly, &c.

Hence $z = \frac{8}{m}x^{\frac{3}{2}} - \frac{24}{m^2}x^2 + \frac{96}{5m^3}x^{\frac{5}{2}} + \frac{64}{3m^4}x^3$ &c.

Or $z = \frac{8}{m}x^{\frac{3}{2}}$ nearly.

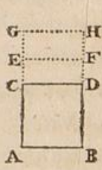
But, by the first process, when $x = 3$, $z = \cdot 886$; which substituted for them, we have $z = \cdot 886$, and the series $=$ $1 \cdot 63$; therefore the correct fluents are

$$z - \cdot 886 = -1 \cdot 63 + \frac{8}{m}x^{\frac{3}{2}} - \frac{24}{m^2}x^2 \text{ \&c,}$$

$$\text{or } z + \cdot 744 = \frac{8}{m}x^{\frac{3}{2}} - \frac{24}{m^2}x^2 \text{ \&c.}$$

And when $z = 3 = \text{AC}$, it gives $x = 6 \cdot 369$ for the height of the tide without, when the ditches are filled to the top of the sluice, or 3 feet high; which answers to $3^{\text{h}}\ 11'\ 4''$.

Lastly, to find the time of rising the remaining 3 feet above the top of the sluice; let



$x = \text{CG}$ the height of the tide above CD,
$z = \text{CE}$ ditto in the ditches above CD;
and the other dimensions as before.
Then $\sqrt{g} : \sqrt{\text{EG}} :: 2g : 2\sqrt{g(x-z)} =$ the velocity with which the water runs through the whole sluice AD; conseq. $\text{AD} \times 2\sqrt{g(x-z)} =$ $18\sqrt{g(x-z)}$ is the quantity per second running through the sluice, and $\frac{18\sqrt{g}}{\text{A}}\sqrt{(x-z)} = v$ the velocity of z, or the rise of the water in the ditches, per second; hence $v : \dot{z} :: 1'' :$ $\dot{t} = \frac{\dot{z}}{v} = \frac{\text{A}}{18\sqrt{g}} \times \frac{\dot{z}}{\sqrt{(x-z)}} = 1800\dot{x}$, and $m\dot{z} = \dot{x}\sqrt{(x-z)}$ is the fluxional equation; where $m = \frac{\text{A}}{1800\sqrt{g}} = \frac{3200}{2079}$.

To find the fluent,

Assume $z = \text{A}x^{\frac{3}{2}} + \text{B}x^{\frac{4}{2}} + \text{C}x^{\frac{5}{2}} + \text{D}x^{\frac{6}{2}}$ &c.

Then $x - z = x - \text{A}x^{\frac{3}{2}} - \text{B}x^{\frac{4}{2}} - \text{C}x^{\frac{5}{2}}$ &c,

$$\dot{x}\sqrt{(x-z)} = x^{\frac{1}{2}}\dot{x} - \frac{A}{2}x^{\frac{2}{2}}\dot{x} - \frac{A^2+4B}{8}x^{\frac{3}{2}}\dot{x} \text{ \&c.}$$

$$m\dot{z} = \tfrac{3}{2}nAx^{\frac{1}{2}}\dot{x} + \tfrac{4}{2}nBx^{\frac{2}{2}}\dot{x} + \tfrac{5}{2}nCx^{\frac{3}{2}}\dot{x} \text{ \&c.}$$

Then equating the like terms gives

$$A = \frac{2}{3n},\ B = \frac{-1}{6n^2},\ C = \frac{1}{90n^3},\ D = \frac{-1}{810n^4}, \text{ \&c.}$$

Hence $z = \frac{2}{3n}x^{\frac{3}{2}} - \frac{1}{6n^2}x^2 + \frac{1}{90n^3}x^{\frac{5}{2}} - \frac{1}{810n^4}x^3$ &c.

But by the second case, when $z = 0$, $x = 3{\cdot}369$, which being used in the series, it is $1{\cdot}936$; therefore the correct fluent is $z = -1{\cdot}936 + \frac{2}{3n}x^{\frac{3}{2}} - \frac{1}{6n^2}x^2$ &c. And when $z = 3$, $x = 7$; the heights above the top of the sluice, answering to 6 and 10 feet above the bottom of the ditches. That is, for the water to rise to the height of 6 feet within the ditches, it is necessary for the tide to rise to 10 feet without, which just answers to 5 hours; and so long it would take to fill the ditches 6 feet deep with water, their horizontal area being 200000 square feet.

Further, when $x = 6$, then $z = 2{\cdot}117$ the height above the top of the sluice; to which add 3, the height of the sluice, and the sum $5{\cdot}117$, is the depth of water in the ditches in 4 hours and a half, or when the tide has risen to the height of 9 feet without the ditches.

Note. In the foregoing problems, concerning the efflux of water, it is taken for granted that the velocity is the same as that which is due to the whole height of the surface of the supplying water: a supposition which agrees with the principles of the greater number of authors: though some make the velocity to be that which is due to the half height only: and others make it still less.

Also in some places, where the difference between two parabolic segments was to be taken, in estimating the mean velocity of the water through a variable orifice, I have used a near mean value of the expression; which makes the operation of finding the fluents much more easy, and is at the same time sufficiently exact for the purpose in hand.

We may further add a remark here concerning the method of finding the fluents of the three fluxional forms that occur in the solution of this problem, viz, the three forms $m\dot{z} = (2x + z)\sqrt{(x - z)}\dot{x}$, and $m\dot{z} = (12 + z)\sqrt{(x - z)}\dot{x}$, and $m\dot{z} = \sqrt{(x - z)}\dot{x}$, the fluents of which are found by assuming the fluent mz in an infinite series ascending in terms of x with indeterminate coefficients A, B, C, &c, which coefficients are afterwards determined in the usual way, by equating the corresponding terms of two similar and equal series, the one series denoting one side of the fluxional equation, and the other series the other side. By similar series, is meant such as have equal or like exponents; though it is not necessary that the exponents of all the terms should be like or pairs, but only some of them, as those that are not in pairs will be cancelled or expelled by making their coefficients $= 0$ or nothing. Now the general way to make the two series similar, is to assume the fluent z equal to a series in terms of x, either ascending or descending, as here

$$z = x^r + x^{r+s} + x^{r+2s} \text{ \&c for ascending,}$$

$$\text{or } z = x^r + x^{r-s} + x^{r-2s} \text{ \&c for a descending}$$

series, having the exponents r, $r \pm s$, $r \pm 2s$, &c, in arithmetical progression, the first term r, and common difference s; without the general coefficients A, B, C, &c, till the values of the exponents be determined. In terms of this assumed series for z, find the values of the two sides of the given fluxional equation, by substituting in it the said series instead of z; then put the exponent of the first term of the one side equal that of the other, which will give the value of the first exponent r; in like manner put the exponents of the two 2d terms equal, which will give the value of the common difference s; and hence the whole series of exponents r, $r \pm s$, $r \pm 2s$, &c, becomes known.

Thus, for the last of the three fluxional equations above mentioned, viz, $m\dot{z} = \sqrt{(x-z)}\dot{x}$, or only $\dot{z} = \sqrt{(x - z)}\dot{x}$; having assumed as above $z = x^r + x^{r+s}$ &c, and taking the fluxion, then $\dot{z} = x^{r-1}\dot{x} + x^{r+s-1}\dot{x} +$ &c, omitting the coefficients; and the other side of the equation $\sqrt{(x - z)}\dot{x} =$

$\surd(x - x^r - x^{r+s}$ &c) $= x^{\frac{1}{2}}\dot{x} - x^{r-\frac{1}{2}}\dot{x}$ &c. Now the exponents of the first terms made equal, give $r - 1 = \frac{1}{2}$, theref. $r = 1 + \frac{1}{2} = \frac{3}{2}$; and those of the 2d terms made equal, give $r + s - 1 = r - \frac{1}{2}$, theref. $s - 1 = -\frac{1}{2}$, and $s = 1 - \frac{1}{2} = \frac{1}{2}$; conseq. the whole assumed series of exponents r, $r + s$, $r + 2s$, &c, become $\frac{3}{2}$, $\frac{4}{2}$, $\frac{5}{2}$, &c, as assumed above in pa. 368.

Again, for the 2d equation $m\dot{z}$ or $\dot{z} = (12 + z)\surd(x - z)\dot{x} = (a + z)\surd(x - z)\dot{x}$; assuming $z = x^r + x^{r+s}$ &c as before, then $\dot{z} = x^{r-1}\dot{x} + x^{r+s-1}\dot{x}$ &c, and $\surd(x - z)\dot{x} = x^{\frac{1}{2}}\dot{x} - x^{r-\frac{1}{2}}\dot{x}$ &c, both as above; this mult. by $a + z$ or $a + x^r + x^{r+s}$ &c, gives $ax^{\frac{1}{2}}\dot{x} - ax^{r-\frac{1}{2}}\dot{x}$ &c: then equating the first exponents gives $r - 1 = \frac{1}{2}$ or $r = \frac{3}{2}$, and $r + s - 1 = r - \frac{1}{2}$ or $s = 1 - \frac{1}{2} = \frac{1}{2}$; hence the series of exponents is $\frac{3}{2}$, $\frac{4}{2}$, $\frac{5}{2}$, &c, the same as the former, and as assumed in pa. 366.

Lastly, assuming the same form of series for z and $\dot{z}$ as in the above two cases, for the 1st fluxional equation also, viz, $m\dot{z} = (2x + z)\surd(x - z)\dot{x}$: then $\surd(x - z)\dot{x} = x^{\frac{1}{2}}\dot{x} - x^{r-\frac{1}{2}}\dot{x}$ &c, which mult. by $2x + z$, gives $2x^{\frac{3}{2}}\dot{x} - x^{r+\frac{1}{2}}\dot{x}$ &c: here equating the first exponents gives $r - 1 = \frac{3}{2}$, or $r = \frac{5}{2}$; and equating the 2d exponents gives $r + s - 1 = r + \frac{1}{2}$, or $s = \frac{3}{2}$; hence the series of exponents in this case is $\frac{5}{2}$, $\frac{8}{2}$, $\frac{11}{2}$, &c, as used for this case in pa. 365. Then, in every case, the general coefficients A, B, C, &c, are joined to the assumed terms x^r, x^{r+s}, &c, and the whole process conducted as in the three pages just referred to.

Such then is the regular and legitimate way of proceeding, to obtain the form of the series with respect to the exponents of the terms. But, in many cases we may perceive at sight, without that formal process, what the law of the exponents will be, as I indeed did in the solutions in the pages above referred to; and any person with a little practice may easily do the same.

PROBLEM XXXI.

To determine the Fall of the Water under the Arches of a Bridge.

The effects of obstacles placed in a current of water, such as the piers of a bridge, are, a sudden steep descent, and an increase of velocity in the stream of water, just under the arches, more or less in proportion to the quantity of the obstruction and velocity of the current: being very small and hardly perceptible where the arches are large and the piers few or small, but in a high and extraordinary degree at London-bridge, and some others, where the piers and the sterlings are so very large, in proportion to the arches. This is the case, not only in such streams as run always the same way, but in tide rivers also, both upward and downward, but much less in the former than in the latter. During the time of flood, when the tide is flowing upward, the rise of the water is against the under side of the piers; but the difference between the two sides gradually diminishes as the tide flows less rapidly towards the concluson of the flood. When this has attained its full height, and there is no longer any current, but a stillness prevails in the water for a short time, the surface assumes an equal level, both above and below bridge. But, as soon as the tide begins to ebb or return again, the resistance of the piers against the stream, and the contraction of the waterway, cause a rise of the surface above and under the arches, with a fall and a more rapid descent in the contracted stream just below. The quantity of this rise, and of the consequent velocity below, keep both gradually increasing, as the tide continues ebbing, till at quite low water, when the stream or natural current being the quickest, the fall under the arches is the greatest. And it is the quantity of this fall which it is the object of this problem to determine.

Now, the motion of free running water is the consequence of, and produced by the force of gravity, as well as that of any other falling body. Hence the height due to the velocity, that is, the height to be freely fallen by any body to acquire the observed velocity of the natural stream, in the river a little way above bridge, becomes known. From the same velocity also will be found that of the increased current in the narrowed way of the arches, by taking it in the reciprocal proportion of the breadth of the river above, to the contracted way in the arches; viz, by saying, as the latter is to the former, so is the first velocity, or slower motion, to the quicker. Next, from this last velocity, will be found the height due to it as before, that is, the height to be freely fallen through by gravity, to produce it. Then the difference of these two heights, thus freely fallen by gravity, to produce the two velocities, is the required quantity of the waterfall in the arches; allowing however, in the calculation, for the contraction, in the narrowed passage, at the rate as observed by Sir I. Newton, in prop. 36 of the 2d book of the Principia, or by other authors, being nearly in the ratio of 25 to 21, or sufficiently near as 24 to 20, that is, as 6 to 5.

Such then are the elements and principles, on which the solution of the problem is easily made out as follows.

Let b = the breadth of the channel in feet;
v = mean velocity of the water in feet per second;
c = breadth of the waterway between the obstacles.

Now $6 : 5 :: c : \frac{5}{6}c$, the waterway contracted as above.

And $\frac{5}{6}c : b :: v : \frac{6b}{5c}v$, the velocity in the contracted way.

Also $32^2 : v^2 :: 16 : \frac{1}{64}v^2$, height fallen to gain the velocity v.

And $32^2 : (\frac{6b}{5c}v)^2 :: 16 : (\frac{6b}{5c})^2 \times \frac{1}{64}v^2$, ditto for the vel. $\frac{6b}{5c}v$.

Then $(\frac{6b}{5c})^2 \times \frac{v^2}{64} - \frac{v^2}{64}$ is the measure of the fall required.

Or $[(\frac{6b}{5c})^2 - 1] \times \frac{vv}{64}$ is a rule for computing the fall.

Or rather $\frac{1 \cdot 44b^2 - c^2}{64c^2} \times v^2$ very nearly, for the fall.

EXAM. 1. *For London-bridge.*

By the observations made by Mr. Labelye in 1746,
The breadth of the Thames at London-bridge is 926 feet;
The sum of the waterways at the time of low-water is 236 ft.;
Mean velocity of the stream just above bridge is $3\frac{1}{6}$ ft. per sec.
But under almost all the arches are driven into the bed great numbers of what are called dripshot piles, to prevent the bed from being washed away by the fall. These dripshot piles still further contract the waterways, at least $\frac{1}{6}$ of their measured breadth, or near 39 feet in the whole; so that the waterway will be reduced to 197 feet, or in round numbers suppose 200 feet.

Then $b = 926$, $c = 200$, $v = 3\frac{1}{6} = \frac{19}{6}$.

Hence $\frac{1{\cdot}44b^2 - c^2}{64c^2} = \frac{1234765 - 40000}{64 \times 40000} = {\cdot}467$.

And $v^2 = \frac{19^2}{6^2} = 10\frac{1}{36}$.

Theref. ${\cdot}467 \times 10\frac{1}{36} = 4{\cdot}63$ ft. $=$ 4 ft. $8\frac{1}{5}$ in. the fall required. By the most exact observations made about the year 1736, the measure of the fall was 4 feet 9 inches.

EXAM. 2. *For Westminster-bridge.*

Though the breadth of the river at Westminster-bridge is 1220 feet; yet, at the time of the greatest fall, there is water through only the 13 large arches, which amount to but 820 feet; to which adding the breadth of the 12 intermediate piers, equal to 174 feet, gives 994 for the breadth of the river at that time; and the velocity of the water a little above the bridge, from many experiments, is not more $2\frac{1}{4}$ feet per second.

Here then $b = 994$, $c = 820$, $v = 2\frac{1}{4} = \frac{9}{4}$.

Hence $\frac{1{\cdot}44b^2 - c^2}{64c^2} = \frac{1422771 - 672400}{64 \times 672400} = {\cdot}017436$.

And $v^2 = \frac{81^2}{16} = 5\frac{1}{16}$.

Theref. ${\cdot}017436 \times 5\frac{1}{16} = {\cdot}08827$ ft. $=$ 1 in. the fall required;

which is about half an inch more than the greatest fall observed by Mr. Labelye.

And, for Blackfriar's-bridge, the fall will be much the same as that of Westminster.

*** See other solutions at pa. 87, &c, vol. 1.

PROBLEM XXXII.

To determine the Circumstances relating to the Pressure of the Atmosphere on a given Space on the Earth, with the Height of a uniform Atmosphere, &c.

It is a fact, and may easily be shown, that the height is a constant quantity, or always the same, of a uniform atmosphere above any place, which shall be all of the uniform density with the air there, at any time, and whatever the weight of it then be as measured by the barometer. This property happens from the circumstance of the density of the air, at the earth's surface, always varying in proportion as indicated by the barometer; in fact, the height of the barometer at once shows both the density of the air and its weight, at the time. So that, as the density varies in exact proportion to the weight of the column, it therefore requires a column of the same height, in all cases, to make the respective weights or pressures; when the temperature or heat of the air is the same. Thus, if W and w denote the weights of atmosphere above any places, D and d their respective densities, and H, h the heights of the uniform columns, of these densities and weights: Then, each density multiplied by the height being equal to the weight, viz. $H \times D = W$, and $h \times d = w$; hence $\frac{W}{D} = H$, and $\frac{w}{d} = h$; but, as the weight is always proportional to the density, therefore $\frac{W}{D} = \frac{w}{d}$, that is, $H = h$, or the heights are equal.

The general height of the uniform atmospheric column is thus easily found. The weight of a cubic foot of mercury is 13600 ounces, or the pressure of a column of mercury 1 foot in height on a square foot of base; but the medium pressure of the atmosphere is equal to the column of 29·75 inches of the barometer; therefore 12 : 29·75 : : 13600 : 33717 ounces pressure of the atmosphere on a square foot; hence $33717 \div 144 = 234$ ounces or $14\frac{5}{8}$ lb. is the medium pressure of the atmosphere on a square inch.

Again, for the height of a uniform atmosphere. Since 33717 ounces is the pressure of the atmosphere on a square foot; and $1\frac{2}{9}$ ounces is the weight of a cubic foot of air, or its pressure on a square foot of 1 foot in height; therefore $1\frac{2}{9}$: 1 : : 33717 : 27600 feet the height the atmosphere would be if it were all of the same uniform density as at the earth, being equal to $5\frac{1}{4}$ miles in height very nearly.

PROBLEM XXXIII.

To show that the Density of the Atmosphere, at different Heights above the Earth, Decreases in such Sort, that when the Heights Increase in Arithmetical Progression, the Densities Decrease in Geometrical Progression.

Let the indefinite perpendicular line AP, erected on the earth, be conceived to be divided into a great number of very small equal parts, A, B, C, D, &c, forming so many thin strata of air in the atmosphere, all of different density, gradually decreasing from the greatest at A: then the density of the several strata A, B, C, D, &c, will be in geometrical progression decreasing.

For, as the strata A, B, C, &c, are all of equal thickness, the quantity of matter in each of them, is as the density there; but the density in any one, being as the compressing force, is as the weight or quantity of all the matter from that place upward to the top of the atmosphere; therefore the quantity of matter in each stratum, is also as the whole quantity from that place upward. Now, if from the whole weight at any place as B, the weight or quantity in the stratum B be subtracted, the remainder is the weight at the next stratum C; that is, from each weight subtracting a part which is proportional to itself, leaves the next weight; or, which is the same thing, from each density subtracting a part which is proportional to itself, leaves the next density. But when any quantities are continually diminished by parts which are proportional to themselves, the remainders form a series of continued proportionals: consequently these densities are in geometrical progression.

Thus, if the first density be D, and from each be taken its nth part; there will then remain its $\frac{n-1}{n}$ part, or the $\frac{m}{n}$ part, putting m for $n-1$; and therefore the series of densities will be D, $\frac{m}{n}$D, $\frac{m^2}{n^2}$D, $\frac{m^3}{n^3}$D, $\frac{m^4}{n^4}$D, &c, the common ratio of the series being that of n to m.

SCHOLIUM.

Because the terms of an arithmetical series, are proportional to the logarithms of the terms of a geometrical series: therefore different altitudes above the earth's surface, are as the logarithms of the densities, or of the weights of air, at those altitudes.

So that, if D denote the density at the altitude A,
and d - the density at the altitude a;
then A being as the log. of D, and a as the log. of d,
the dif. of alt. A $-a$ will be as the log. D $-$ log. d. or log. $\frac{D}{d}$.

And if A$=0$, or D the density at the surface of the earth; then any altitude above the surface a, is as the log. of $\frac{D}{d}$.

Or, in general, the log. of $\frac{D}{d}$ is as the altitude of the one place above the other, whether the lower place be at the surface of the earth, or any where else.

And from this property is derived the method of determining the heights of mountains and other eminences, by the barometer. For, by taking, with this instrument, the pressure or density, at the foot of a hill for instance, and again at the top of it, the difference of the logarithms of these two pressures, or the logarithm of their quotient, will be as the difference of altitude, or as the height of the hill; supposing the temperatures of the air to be the same at both places, and the gravity of air not altered by the different distances from the earth's centre.

But as this formula expresses only the relations between different altitudes with respect to their densities, recourse must be had to some experiment, to obtain the real altitude which corresponds to any given density, or the density which corresponds to a given altitude. And there are various experiments by which this may be done. The first, and most natural, is that which results from the known specific gravity of air, with respect to the whole pressure of the atmosphere on the surface of the earth. Now, as the altitude a is always as log. $\frac{D}{d}$; assume h so as that $a = h \times \log.\frac{D}{d}$, where h will be of one constant value for all altitudes; and to determine that value, let a case be taken in which we know the altitude a corresponding to a known density d; as for instance, take $a = 1$ foot, or 1 inch, or some such small altitude; then, because the density D may be measured by the pressure of the atmosphere, or the uniform column of 27600 feet, when the temperature is 55°; therefore 27600 feet will denote the density D at the lower place, and 27599 the less density d at 1 foot above it; consequently $1 = h \times \log.\frac{27600}{27599}$; which, by the nature of logarithms, is nearly $= h \times \frac{\cdot 43429448}{27600}$ $= \frac{h}{63551}$ nearly; and hence $h = 63551$ feet; which gives, for any altitude in general, this theorem, viz. $a = 63551 \times \log.\frac{D}{d}$, or $= 63551 \times \log.\frac{M}{m}$ feet, or $10592 \times \log.\frac{M}{m}$ fathoms; where M is the column of mercury which is equal to the pressure or weight of the atmosphere at the bottom, and m that at the top of the altitude a; and where M and m may be taken in any measure, either feet or inches, &c.

Note, that this formula is adapted to the mean temperature of the air 55°. But, for every degree of temperature different from this, in the medium between the temperatures at the top and bottom of the altitude a, that altitude will vary by its 435th part; which must be added, when that medium exceeds 55°, otherwise subtracted.—Note, also, that a column of 30 inches of mercury varies its length by about the $\frac{1}{320}$ part of an inch for every degree of heat, or rather $\frac{1}{9600}$ of the whole volume.

But the formula may be rendered much more convenient for use, by reducing the factor 10592 to 10000, by changing the temperature proportionally from 55°. Thus, as the diff. 592 is the 18th part of the whole factor 10592; and as 18 is the 24th part of 435; therefore the correspondent change of temperature is 24°, which reduces the 55° to 31°. So that the formula is, $a = 10000 \times \log. \frac{M}{m}$ fathoms, when the temperature is 31 degrees; and for every degree above that, the result is to be increased by so many times its 435th part.

PROBLEM XXXIV.

To divide a Given Circle into any proposed number of Equal Parts, by means of other Circles Concentric with the Given one.

This problem is now added here in the appendix, having been omitted in its proper place, Tract xiv. vol. 1, beside another problem, allied to this, as well in their nature as in their fate and consequences.

A particular case of the present problem was first of all, as far as I know, proposed in that useful and valuable little annual work, the Ladies' Diary, for the year 1709, in this form, viz, Seven men bought a grinding-stone, of 5 feet or 60 inches in diameter: and they agreed together, that each should grind off an equal share; so that one, beginning first, should grind his 7th part off the stone; then a second should grind *his;* and so continue in succession. The question then was, how much of the diameter must each person grind down. In the year following an answer was so far given only, as merely to specify the numbers denoting the parts of the diameter to be ground down, or cut off, without any mode of solution whatever, either arithmetical or geometrical.

Many years after, a geometrical construction was given by a Mr. Hawney, in his little book on Mensuration, but so clumsy in its manner, as to require the description of a separate circle to ascertain the point through which each of the dividing concentric circles was to pass. And in this state it remained till about the year 1770, when Mr. James Ferguson, the ingenious lecturer on astronomy and mechanics, in his peregrinations came to Newcastle, where I then resided, to give the usual course of his public lectures; on which occasion, with the assistance of my friends, I not only procured him a numerous and respectable audience, but also accommodated him with the free use of the new school-rooms, which I had lately built, to deliver his lectures in. As Mr. F. commonly amused my family and friends at evenings, with showing his ingenious mechanical contrivances and drawings, on one of these occasions he produced a very neat and correct drawing, on a large scale, being a construction of this problem, in the very prolix way as before given by Hawney, but which he exhibited as a great curiosity. I ventured to remark to him that I thought a much simpler construction might be found out, for this problem, which was then new to me. As Mr. F. expressed a wish to see such a thing as a simpler construction, which however he seemed to have his doubts of procuring, I was induced to consider it that evening, before going to rest, and discovered the construction as follows.

The next morning I showed him the new and very simple construction, with its demonstration, which he seemed much pleased with, on account of the apparent simplicity, but doubted very much that it might not be correctly true.

On referring him to the accompanying demonstration, to satisfy himself of its geometrical truth, I was much surprized by his reply, that he could not understand that, but he would make the drawing correctly on a large scale, which was always his way to try if such things were true. In my surprize I asked where he had learned geometry, and by what Euclid or other book; to which he frankly replied he had never learned any geometry, nor could ever understand the demonstration of any one of Euclid's propositions. Accordingly the next morning, with a joyful countenance, he brought me the construction, neatly drawn out on a large sheet of pasteboard, saying he esteemed it a treasure, having found it quite right, as every point and line agreed to a hair's breadth, by measurement on the scale. This problem and the construction he afterwards inserted, with the proper acknowledgment, as a curiosity, in his Select Mechanical Exercises, p. 123, printed in 1773.

Also, in the beginning of the year 1771, when I commenced the republication of the Ladies' Diary Questions, I inserted the following solution of the question, at pa. 53 of the first volume of that work, viz.

"This question is to divide a circle, of 60 inches diameter, into 7 equal parts, or rings, bounded by concentric circles; of which the solution will be thus.—The whole circle, and each inner circle, after the several preceding rings are ground off, must be to each other, by the question, as the numbers 7, 6, 5, 4, 3, 2, 1; but circles are as the squares of their diameters; therefore the diameters of those circles will be to one another as $\sqrt{7}$, $\sqrt{6}$, $\sqrt{5}$, $\sqrt{4}$, $\sqrt{3}$, $\sqrt{2}$, $\sqrt{1}$: but the greatest diameter is 60 or $60\sqrt{\frac{7}{7}}$; therefore, by proportioning, all the other diameters will come out thus, $60\sqrt{\frac{6}{7}}$, $60\sqrt{\frac{5}{7}}$, $60\sqrt{\frac{4}{7}}$, $60\sqrt{\frac{3}{7}}$, $60\sqrt{\frac{2}{7}}$, $60\sqrt{\frac{1}{7}}$: Now the last of these is the diameter of the last person's share, and the difference between every two adjacent terms being taken, will give the double breadth of the rings, or the parts of the whole diameter to be ground off by the other persons; viz.

$$60\sqrt{\tfrac{7}{7}}-60\sqrt{\tfrac{6}{7}}=\frac{\sqrt{7}-\sqrt{6}}{\sqrt{7}}\times 60=4{\cdot}450794 \text{ the 1st person's share}$$

$$60\sqrt{\tfrac{6}{7}}-60\sqrt{\tfrac{5}{7}}=\frac{\sqrt{6}-\sqrt{5}}{\sqrt{7}}\times 60=4{\cdot}839951 \text{ the 2d}$$

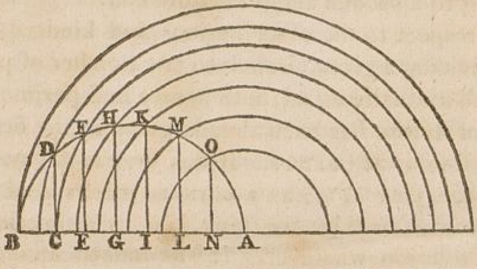
$$60\sqrt{\tfrac{5}{7}}-60\sqrt{\tfrac{4}{7}}=\frac{\sqrt{5}-\sqrt{4}}{\sqrt{7}}\times 60=5{\cdot}353518 \text{ the 3d}$$

$$60\sqrt{\tfrac{4}{7}}-60\sqrt{\tfrac{3}{7}}=\frac{\sqrt{4}-\sqrt{3}}{\sqrt{7}}\times 60=6{\cdot}076516 \text{ the 4th}$$

$$60\sqrt{\tfrac{3}{7}}-60\sqrt{\tfrac{2}{7}}=\frac{\sqrt{3}-\sqrt{2}}{\sqrt{7}}\times 60=7{\cdot}207871 \text{ the 5th}$$

$$60\sqrt{\tfrac{2}{7}}-60\sqrt{\tfrac{1}{7}}=\frac{\sqrt{2}-\sqrt{1}}{\sqrt{7}}\times 60=9{\cdot}393480 \text{ the 6th}$$

$$60\sqrt{\tfrac{1}{7}}-60\sqrt{\tfrac{0}{7}}=\frac{1}{\sqrt{7}}\times 60=22{\cdot}677870 \text{ the 7th}$$



"The Construction will be thus.—Divide the radius AB of the given circle into 7 equal parts; and at the points of division erect perpendiculars meeting the circle, described on AB as a diameter in D, F, H, K, M, O; then with the centre A, and radii AO, AM, AK, &c, circles being described, the thing is done.—For, by the nature of the circle, the squares of the chords or radii, AO, AM, AK, &c, are as the versed sines AN, AL, AI, &c.

"Scholium. It is evident that the above method of calculation and construction will both hold true also when the shares are unequal in any proportion; by using the respective proportional numbers in the former, and dividing the radius AB in the same proportion in the latter.

Very soon after the first publication of the solution of the Diary question as above, it was seized by a shameful plagiary, in the person of a Mr. Samuel Clark, who had commenced, after mine, a republication of the Diary questions, under the title of 'The Diarian Repository.' When this editor arrived at the question above referred to, he copied my solution, and very inconsistently ascribed it to Mr. Hawney before mentioned, though it was manifest to every geometrician that no two constructions could be more unlike one another, as the one employed seven different circles, and as many separate constructions, in effecting what the other accomplished by only one single circle and construction. From the many gross errors, and the numerous omissions and absurdities in that 'Repository of Errors,' as it was commonly called, it soon fell under the necessity of coming to a sudden and premature end.

With respect to the other curious and kindred problem, that of dividing a given circle into any number of parts, that may be all mutually equal, both in area and perimeter, some account of its rise has been already given in the first volume of these tracts, at pa. 254. It was first anonymously proposed in the year 1774, as a curious paradoxical problem, but unaccompanied by the least hint or intimation of any mode of solution whatever. It was indeed announced by the proposer expressly as a seeming paradox, but accompanied with the declaration that it nevertheless was capable of a strict geometrical solution. The problem remained however some time unanswered, being given up by all persons as a matter quite hopeless, and by most deemed in fact as little to be expected as the quadrature of the circle itself, to which it was thought to be nearly allied, and indeed dependent on it; for no person could imagine any other possible way of a circle being divided, even in idea, into any *number* of such parts, that might be equal both in area and perimeter, than by radii drawn from the centre to the points of equal divisions in the circumference. This was, in effect, reducing the problem to this other, of dividing the circumference in *any* proposed number of equal parts, which was deemed on all hands a thing impossible to be effected. After some time no person thought any more of the matter, but as a thing never to be accomplished; and so I believe it might have remained to this day, but for the occurrence of some such accident as that which actually led myself into the train of thought which soon ended in the complete solution. The construction I first inserted in the Critical Review, as before mentioned in the first volume of these Tracts; next it was introduced into my first or quarto volume of Tracts, published in the year 1786, accompanied with a short account of its rise, and a considerable improvement of it, by rendering the property general for the division into all ratios of parts, equal or unequal, and extending the same to all ellipses, as well as circles. After which I have usually been in the habit of introducing it into my Dictionary, and the more common elementary books on mensuration, &c.

Lastly, finding the two constructions introduced, by my friend Mr. Leslie, the ingenious and learned mathematical professor in the university of Edinburgh, into the first edition of his Geometry, published in 1809, both together in pages 222 and 223: as these problems were rather of an uncommon nature, I did think some mention might have been made of their origin, or the circumstances that have attended them; and I hinted as much to my ingenious friend. In consequence of which, probably, I find that the learned author has, in the 2nd edition of his work, separated those two constructions, placing one among the elements at p. 181, and the other among the notes at p. 432, accompanied with the note, that it was the result of a 'principle briefly *suggested* by Mr. Lawson, and afterwards explained and demonstrated in Dr. Hutton's Mathematical Tracts.' This change and announce seemed to make the matter rather worse than before, as it appeared less unfriendly, or less uncivil, to omit noticing a fact entirely, than to mis-state it. For, certain it is, that Mr. Lawson never *suggested* any principle or extension, nor any mode of solution whatever; the discovery having been made and published by myself alone.

FINIS.

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